

Co-monotonic default quote paths for basket evaluation

To evaluate structured credit products such as default baskets and collateralised debt obligations, it is essential to fully understand the default timing of credit-risky instruments in the underlying reference portfolio. Christian Bluhm and Ludger Overbeck investigate a simulation scheme applicable to such products based on a ‘one-variable approach’ in the same way as common analytic approximation techniques but applicable to low-granularity portfolios with an arbitrary dependence structure and non-uniform default probabilities

The evaluation of structured credit products such as default baskets and collateralised debt obligations (CDOs) is usually based on a so-called ‘scenario transformation’ approach that addresses the following issues. Simulation of a default times vector for the portfolio underlying the transaction yields a so-called ‘asset-side scenario’ (see the left side in figure 1, where the distribution of loss scenarios of a credit portfolio is shown). Then, every single asset-side scenario is transformed into a so-called ‘liability-side scenario’ by application of the structural definition of the transaction, for example, in case of a CDO described by the offering circular or by the presale report of a rating agency (for a brief overview only). As a last step, all liability-side scenarios are aggregated into certain cashflow, return or loss distributions, for example, the loss distribution for a considered CDO tranche (see, for example, Bluhm & Overbeck, 2004a).

The more complicated the structural definition of a transaction, the more important is the scenario transformation scheme, in order to gain insight about the link between the performance of the underlying asset pool and the performance of securities on the liability side of the structure. For less complicated structures, analytic and semi-analytic approaches have a good chance of approximating a ‘full’ Monte Carlo simulation approach in a satisfactory manner. Analytic and semi-analytic approaches are most often based on analytic portfolio models for the underlying credit portfolio. In the context of latent variable models, the best known analytic credit risk model is the threshold approach underlying the benchmark risk weight formula in the new Basel capital Accord (see Basel Committee on Banking Supervision, 2004, Vasicek, 1991, Finger, 1999, and Gordy, 2000 and 2003). An application of the analytic threshold approach to CDO evaluation and an extension to semi-analytic techniques can be found in Bluhm & Overbeck (2004b).

A major issue in basket or CDO evaluation is the timing of defaults. For example, if in an arbitrage-motivated CDO the occurrence of defaults triggers an excess spread ‘trapping’, then the excess spread collected by equity investors strongly depends on default timing. If defaults occur more frontloaded (that is, at early times), then excess cash is trapped early and used as a credit enhancement for senior tranche investors. Equity note-holders then will not only suffer a capital loss but will also not be able to collect an attractive amount of excess spread. If defaults occur more backloaded (late in time), then equity investors will have time to collect excess spread, compensating them for the capital loss they will suffer. In the case of a backloaded default timing, equity investors may achieve an attractive return on their investment even if they lose their invested capital in part or total.

Now let us introduce some notation in order to become more explicit. Consider a default times vector $\tau = (\tau_1, \dots, \tau_m)$ for a portfolio of m credit-

risky instruments such as loans, bonds or swaps underlying some credit-linked correlation product such as a default basket or a CDO. Every realisation of τ constitutes an asset-side scenario in line with figure 1. The vector τ determines the portfolio’s default quote path $(L_t)_{0 \leq t \leq T}$ until the maturity T of the basket:

$$L_t = \frac{1}{m} \sum_{k=1}^m \mathbf{1}_{\{\tau_k \leq t\}} \quad (1)$$

The inter-temporal dependence of jumps $L_t - L_s$ for times $s < t$, sampled at a discrete time grid $0 = t_0 < t_1 < \dots < t_{q-1} < t_q = T$, is hidden in the joint distribution of the default quote path evaluated at these times. According to Sklar’s theorem (Sklar, 1959 and 1973), the joint distribution of the time-discrete default quote path can be written by means of a copula function:

$$C : [0,1]^q \rightarrow [0,1], (u_1, \dots, u_q) \mapsto C(u_1, \dots, u_q)$$

in the following form:

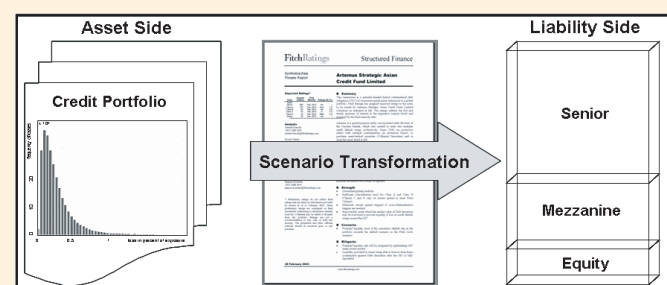
$$\mathbb{P}[L_{t_1} \leq x_1, \dots, L_{t_q} \leq x_q] = C(\mathbb{P}[L_{t_1} \leq x_1], \dots, \mathbb{P}[L_{t_q} \leq x_q]) \quad (2)$$

Note that the points x_i can always be written as k_i/m with $k_i \in \{1, \dots, m\}$ and that, in contrast with the case of continuous distributions, the copula C is not uniquely determined by the joint distribution function of the default quote path.

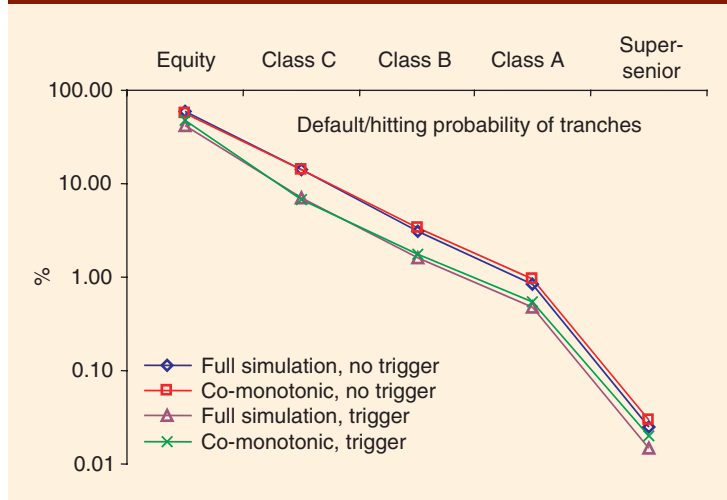
□ **Definition 1.** A default quote path is called co-monotonic if it can be written in the form (2) with $C = \tilde{C}$ being the co-monotonic copula defined by:

$$\tilde{C}(u_1, \dots, u_q) = \min\{u_1, \dots, u_q\} \quad (u_1, \dots, u_q \in [0,1])$$

1. Scenario transformation in a CDO simulation model



2. Comparison of co-monotonic and 'full' Monte Carlo simulation results for a CSO with five tranches



Note that the co-monotonic copula is sometimes called the upper Fréchet copula. It is the copula inducing the strongest possible multivariate dependency.

We point out that for given known marginal default quote variables L_{t_1}, \dots, L_{t_q} the co-monotonic copula always leads to the same distribution of the time until the m th default as any other copula. As a 'side product', we develop an efficient scheme for simulating CDOs and default baskets, for example, the first-to-default, second-to-default, etc. The simulation scheme yields an exact replication of a full Monte Carlo simulation if in the underlying portfolio all assets have the same exposure (money 'at risk'). In the case of non-exchangeable exposures, co-monotonic simulation can still be exploited as a valuable approximation tool. Additionally, the co-monotonic approach can be combined with a uniform portfolio approximation to further speed up CDO simulations (see figure 2).

Co-monotonicity in default quote paths

Co-monotonicity provides a valuable and efficient tool to evaluate the default quote paths of any given credit portfolio model. We will explain this here and in the subsequent sections. For this purpose, let us choose any credit portfolio model and determine the probabilities:

$$p_{t,k} = \mathbb{P}[L_t = k/m] \quad (3)$$

for a portfolio of m names where L_t is defined in equation (1). There are lucky cases where the portfolio model allows us to determine the likelihoods $p_{t,k}$ numerically (see below, 'Implications, benefits and applications of co-monotonicity'). In other cases, for example, in case of a complicated underlying multi-factor model, these likelihoods have to be determined by simulation techniques. In this section, it is essential that these likelihoods are known.

Given the probabilities (3) for a discrete time grid $t = t_1, \dots, t_q$:

$$(p_{t,k})_{t=t_1, \dots, t_q; k=0,1,2, \dots, m} \in \mathbb{R}^{q \times (m+1)}$$

we write $[z] = \max\{k \in \mathbb{N}_0 : k \leq z\}$ for the greatest non-negative integer below $z \geq 0$ and define for any time $t_j > 0$ a distribution function:

$$G_{t_j}(x) = \sum_{k=0}^{[mx]} p_{t_j,k} \quad (x \in [0,1]) \quad (4)$$

The distribution functions G_{t_j} are step functions due to the finite granularity m of the portfolio. We now compare the original default quote path:

$$\mathbf{L} = (L_{t_1}, \dots, L_{t_q}) = \left(\frac{1}{m} \sum_{k=1}^m \mathbf{1}_{\{\tau_k \leq t_1\}}, \dots, \frac{1}{m} \sum_{k=1}^m \mathbf{1}_{\{\tau_k \leq t_q\}} \right)$$

determined by the chosen portfolio model applied in equation (1) with the default quote path:

$$\tilde{\mathbf{L}} = (\tilde{L}_{t_1}, \dots, \tilde{L}_{t_q}) = (G_{t_1}^{-1}(Z), \dots, G_{t_q}^{-1}(Z))$$

defined by the distribution functions (4) where $Z \sim U[0, 1]$ is a random variable uniformly distributed in $[0, 1]$ and:

$$G_{t_j}^{-1}(z) = \inf \left\{ q \geq 0 \mid G_{t_j}(q) \geq z \right\}$$

denotes the generalised inverse of G_{t_j} .

□ **Proposition 1.** The one-dimensional marginal distributions of \mathbf{L} and $\tilde{\mathbf{L}}$ coincide, and $\tilde{\mathbf{L}}$ is co-monotonic.

■ **Proof.** For the coincidence of marginal distributions, consider for $Z \sim U[0, 1]$:

$$\mathbb{P} \left[G_{t_j}^{-1}(Z) \leq \frac{k}{m} \right] = \mathbb{P} \left[Z \leq G_{t_j} \left(\frac{k}{m} \right) \right] = G_{t_j} \left(\frac{k}{m} \right) = \sum_{i=0}^k p_{t_j,i} = \mathbb{P} \left[L_{t_j} \leq \frac{k}{m} \right]$$

due to equations (3) and (4). Co-monotonicity of $\tilde{\mathbf{L}}$ can be seen as follows:

$$\begin{aligned} P \left[G_{t_1}^{-1}(Z) \leq x_{t_1}, \dots, G_{t_q}^{-1}(Z) \leq x_{t_q} \right] &= P \left[Z \leq G_{t_i}(x_{t_i}); t = t_1, \dots, t_q \right] \\ &= \min \left\{ G_{t_i}(x_{t_i}); t = t_1, \dots, t_q \right\} \\ &= \tilde{C} \left(G_{t_1}(x_{t_1}), \dots, G_{t_q}(x_{t_q}) \right) \end{aligned}$$

again due to the uniform distribution of Z in $[0, 1]$. As in definition 1, \tilde{C} denotes the co-monotonic copula.

The coincidence of the marginal distributions of \mathbf{L} and $\tilde{\mathbf{L}}$ can be illustrated by the simulation scheme shown in figure 2, here for a yearly time grid $t_1 = 1, \dots, t_4 = 4$. Co-monotonicity of $\tilde{\mathbf{L}}$ refers to the simulation of just one single random variable $Z \sim U[0, 1]$. Partitioning the unit interval in appropriate pieces to reflect the definition (4) of the discrete distribution functions G_{t_j} yields – by construction – marginal default quote distributions exactly matching the original default quote distributions of L_{t_j} at any time horizon t_j . Therefore, proposition 1 indeed confirms our statement at the beginning of this section that co-monotonicity arises naturally in the context of portfolio default quote paths (independent of the chosen portfolio model).

Co-monotonic default timing

As already indicated, an accurate modelling of the timing of defaults of assets underlying a basket or CDO structure is important. Here, we show that replacing the original copula C in equation (2) by the co-monotonic copula \tilde{C} not only preserves the original marginal default quote distributions (see proposition 1) but also preserves the timing of defaults from the original portfolio model. More explicitly, we show that the co-monotonic default quote path $\tilde{\mathbf{L}}$ generated by means of a simulation scheme as illustrated in figure 3 and the original default quote path \mathbf{L} perfectly agree from an n th-to-default timing perspective.

We denote by $\tau_{n:m}$ the time until the n th default out of m obligors with respect to the original default quote path \mathbf{L} :

$$\tau_{n:m} = \inf \{ t \geq 0 : L_t \geq n/m \}$$

In the co-monotonic set-up, we write $\tilde{\tau}_{n:m}$ for the time of the n th default corresponding to the co-monotonic default quote path $\tilde{\mathbf{L}}$:

$$\tilde{\tau}_{n:m} = \inf \{ t \geq 0 : G_t^{-1}(Z) \geq n/m \} \quad (Z \sim U[0,1])$$

We now show that the distributions of $\tau_{n:m}$ and $\tilde{\tau}_{n:m}$ coincide.

□ **Proposition 2.** The distributions of the n th default time for default quote paths \mathbf{L} and $\tilde{\mathbf{L}}$ agree:

$$\mathbb{P}[\tau_{n:m} \leq t] = \mathbb{P}[\tilde{\tau}_{n:m} \leq t] \quad \text{for all } t \geq 0$$

■ Proof. We start with the co-monotonic path $\tilde{\mathbf{L}}$ and define:

$$q_{t,n:m} = \sum_{k=0}^{n-1} p_{t,k} = \sum_{k=0}^{n-1} \mathbb{P}[L_t = k/m] = \mathbb{P}\left[L_t \leq \frac{n-1}{m}\right] \quad (t \geq 0; n = 1, \dots, m)$$

Note that $t \mapsto q_{t,n:m}$ is a decreasing function (see also figure 3). Based on proposition 1 and figure 3, observing n out of m defaults at time t in a co-monotonic simulation is equivalent to:

$$q_{t,n:m} = G_t\left(\frac{n-1}{m}\right) < Z \leq G_t\left(\frac{n}{m}\right) = q_{t,(n+1):m} \quad (5)$$

Therefore, the time until the n th default must equal the first time t such that $Z > q_{t,n:m}$ for $Z \sim U[0, 1]$:

$$\tilde{\tau}_{n:m} = \inf\{t \geq 0 : Z > q_{t,n:m}\} = \inf\{t \geq 0 : G_t^{-1}(Z) \geq n/m\}$$

We now come to the path \mathbf{L} . In case of $\tau_{n:m} \leq t$ at least n out of m defaults must have occurred before or at time t . The likelihood for n or more defaults in the time interval $[0, t]$ is given by:

$$\begin{aligned} \mathbb{P}[\# \text{ defaults} \in \{n, n+1, \dots, m\} \text{ until time } t] \\ = \sum_{i=n}^m \mathbb{P}[L_t = i/m] = \sum_{i=n}^m p_{t,i} = 1 - q_{t,n:m} \end{aligned}$$

Therefore, we finally obtain the claimed conclusion:

$$\mathbb{P}[\tau_{n:m} \leq t] = \sum_{i=n}^m p_{t,i} = 1 - q_{t,n:m} = \mathbb{P}[Z > q_{t,n:m}] = \mathbb{P}[\tilde{\tau}_{n:m} \leq t]$$

Implications, benefits and applications of co-monotonicity

Proposition 2 says that the distribution of the n th default time can be simulated by a co-monotonic approach according to figure 3, independent of the underlying portfolio model. Now, if all assets in the portfolio have the same amount of money ‘at risk’, there is no loss of information arising from a co-monotonic approach. This is because, based on proposition 2, the co-monotonic approach yields full information regarding the time grid of default occurrences, and the condition of exchangeable exposures (that is, all assets cause the same loss amount in case of default) guarantees that we do not have to be concerned with the question ‘which particular asset is defaulting at which particular time’, because all assets generate the same amount of loss anyway. Fortunately, the condition of ‘exchangeable exposures’ can be assumed to be fulfilled in typical collateralised synthetic obligations and default baskets. It is unusual to have a basket or collateralised swap obligation (CSO) with an inhomogeneous exposure distribution. Therefore, the co-monotonic approach is perfectly suited for such instruments. In ‘classical’ CDOs, say, loan securitisations, the situation is different. Here, the exposure distribution most often is inhomogeneous, although most offering circulars explicitly do not allow for significant single obligor concentrations (in terms of lending amount allocated to a single name). Therefore, the co-monotonic approach can serve as an approximation tool even in such cases.

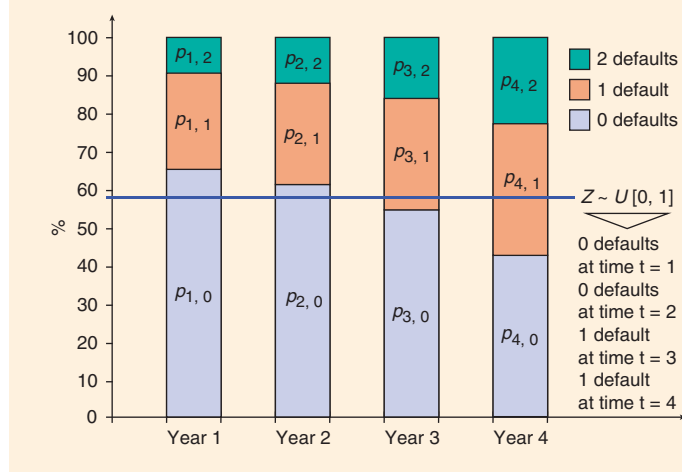
We see two major benefits in the co-monotonic set-up, namely simulation efficiency and variance reduction. The order of magnitude of both benefits depends on the originally chosen model, but even if the original model is a simple one-factor model the benefits are still measurable. For illustration purposes we give some examples. Consider a one-factor model:

$$r_i = \sqrt{\rho}Y + \sqrt{1-\rho}\varepsilon_i \quad (i = 1, \dots, m = \# \text{ obligors}) \quad (6)$$

triggering default indicators $\mathbf{1}_{\{r_i < N^{-1}[p_i^{(i)}]\}}$ leading to default quotes:

$$L_t = \frac{1}{m} \sum_{i=1}^m \mathbf{1}_{\{r_i < N^{-1}[p_i^{(i)}]\}} \quad (7)$$

3. Simulation scheme for $\tilde{\mathbf{L}}$



where $Y, \varepsilon_1, \dots, \varepsilon_m \sim N(0, 1)$ are independent standard normal random variables, $(p_i^{(i)})_{t \geq 0}$ refers to the credit curve¹ of obligor i , ρ denotes the uniform asset correlation referring to the systematic risk of obligors and $N[\cdot]$ denotes the standard normal distribution function. Background information on the one-factor threshold model (before and after application of the limit ‘# obligors $\rightarrow \infty$ ’ respectively), mentioned above, can be found in various publications, for example, Finger (1999), Gordy, 2000 and 2003, Wilde, 2001, Schoenbucher, 2000, and chapter 2 in Bluhm, Overbeck & Wagner (2003). Originally it dates back to a paper by Vasicek (1991).

Comparing (1) and (7) suggests defining the default time of obligor i by:

$$\tau_i = F_i^{-1}(N[r_i]) \quad (i = 1, \dots, m) \quad (8)$$

where r_i is defined in (6) and $F_i(t) = \mathbb{P}[\tau_i \leq t] = p_i^{(i)}$ is the distribution function of the default time on $[0, \infty)$ determined by the corresponding credit curve $(p_i^{(i)})_{t \geq 0}$ (see Finger, 2000, Li, 1999 and 2000, Schmidt & Ward, 2002, and Bluhm & Overbeck, 2004a). The copula driving the tail dependency of default times τ_i according to definition (8) is determined by the copula of the one-factor model (6), which in our case is Gaussian but could as well be defined by means of other copulas, for example, the Student- t copula.

From proposition 2 we know that the time until the n th default out of m obligors based on the original model, $\tau_{n:m}$, coincides in distribution with the corresponding co-monotonic n th default time $\tilde{\tau}_{n:m}$. Based on (6) and (8), it is clear that the simulation of correlated default times (τ_1, \dots, τ_m) for the one-factor model with respect to m names requires the simulation of $(m+1)$ random variables.² In a co-monotonic simulation one has (according to figure 3) only to simulate one single random variable $Z \sim U[0, 1]$.

We tested the simulation time advantage by means of a CSO³ with 80 names (\$10 million each), five tranches (equity, class C, class B, class A and super senior) and typically structured cashflows including an excess spread trigger⁴ in case of losses (see Bluhm & Overbeck, 2004b, for details of the working example). In this example, we observed that the co-monotonic approach was five to ten times faster⁵ than the ‘full’ Monte Carlo simulation approach. However, one has to take into account that we did our test calculations with a one-factor model. If the one-factor model was

¹ A credit curve is a term structure of cumulative default probabilities. See, for example, Bluhm & Overbeck (2004a)

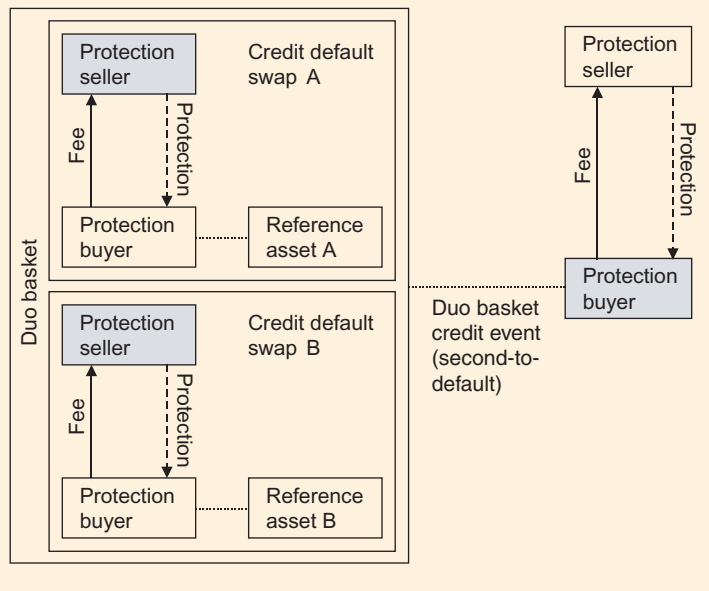
² One systematic factor Y and m idiosyncratic effects $\varepsilon_1, \dots, \varepsilon_m$

³ See, for example, Bluhm & Overbeck (2004a) for further explanations and an example

⁴ The excess spread trigger redirects cash from excess spread into a loss reserve (credit enhancement), protecting senior note investors

⁵ The speed factor depends on several factors, for example, the used computer, the programming environment and so on

4. Duo basket transaction (illustrative)



replaced by a multi-factor model the simulation time advantage could be expected to be significantly higher (when replacing the systematic factor Y by a sum of weighted country and industry factors).

We come to a second example combining the co-monotonic approach with an analytical approximation. We again consider the sample CSO mentioned above. The analytical approximation we use is straightforward and works as follows. We replace the 'true' $p_{t,k}$ s in equation (3) by:

$$p_{t,k} = \mathbb{P}[L_t = k/m]$$

$$= \binom{m}{k} \int_{-\infty}^{\infty} N\left[\frac{N^{-1}(p_t) - \sqrt{\rho}y}{\sqrt{1-\rho}}\right]^k \left(1 - N\left[\frac{N^{-1}(p_t) - \sqrt{\rho}y}{\sqrt{1-\rho}}\right]\right)^{m-k} dN(y) \quad (9)$$

where $(p_t)_{t \geq 0}$ is a uniform⁶ (approximate) term structure for the 80-name portfolio (see the already mentioned literature on one-factor models). In this particular set-up, the probabilities $p_{t,k}$ can conveniently be calculated numerically (for example, by use of Mathematica or other programs that can do numeric integrations).

In figure 2, we compare the hitting probabilities of CSO tranches with respect to a full default times simulation model and a co-monotonic simulation model. One can see that the co-monotonic/analytic approach yields a fairly good approximation of the 'full' simulation model. Based on the additional credit enhancement for senior note investors in case of a default/loss trigger, the hitting probability of tranches in a transaction with an excess spread trigger is significantly lower (see Bluhm & Overbeck, 2004a). If instead of the approximate $p_{t,k}$ s from equation (9) we used the 'true' $p_{t,k}$ s corresponding to the original model, the co-monotonic approximation would turn out to provide a perfect match⁷ based on proposition 2.

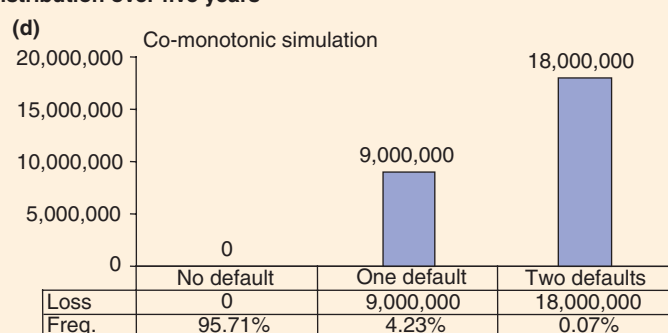
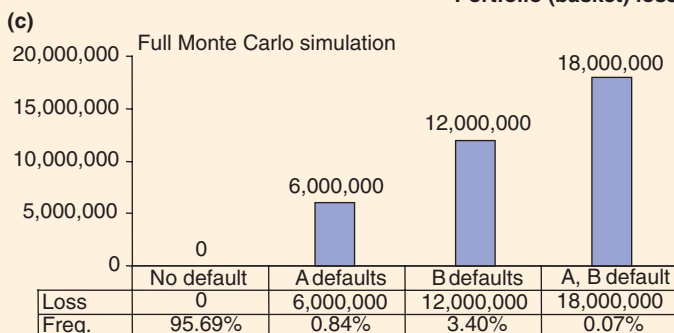
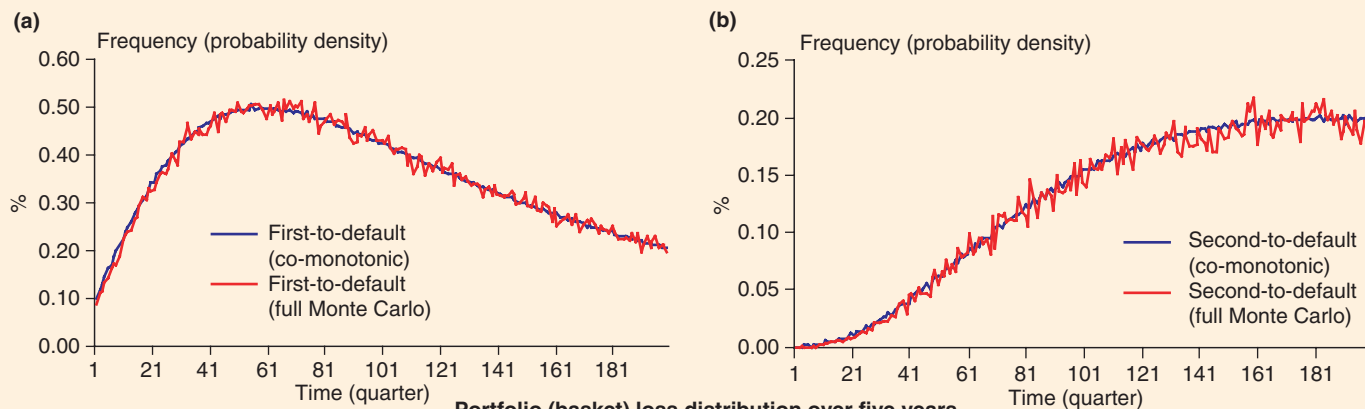
In our last example, we come back to the already mentioned condition of exchangeable exposures. Additionally, we show the variance reduction effect co-monotonicity has on basket simulations. For this purpose, we consider a 'duo basket' (that is, a basket consisting of two assets) containing:

⁶ A uniform credit curve can be obtained as an average of the credit curves of single names in the portfolio

⁷ As already indicated a perfect match is obtained if the realised loss arising from a default is the same no matter what asset defaults

5. Comparison of simulation approaches: efficiency, variance reduction and loss of information

First-to-default and second-to-default distribution of a 'duo basket'



□ instrument A: a five-year CDS written on a name with a one-year default probability of $p_A = 8$ basis points (A-rated) with a volume of €10 million; and

□ instrument B: a five-year CDS written on a name with a one-year default probability of $p_B = 36$ bp (BBB-rated) with a volume of €20 million with an assumed asset correlation of 15%. Duo baskets applied to loans or bonds instead of CDSs can be used to hedge joint defaults in a credit portfolio. Pairing swaps instead of funded instruments synthesises the duo basket transaction towards a purely derivatives-based product. The transaction we have in mind works as follows.

We sell protection on names A and B but buy protection against a basket credit event defined by the second-to-default of the basket. Figure 4 shows the structure of the transaction, where our position is shaded in grey. In other words, we are (synthetically) exposed to the default risk of two names but protected against the joint default of the two names. Based on the moderate asset correlation of A and B we can expect a moderate premium in our protection-buying agreement (right-hand side of figure 4) whereas in our protection-selling agreement (left-hand side of figure 4) we can expect to collect two single-name fees driven by the default risk of assets A ($p_A = 0.0008$) and B ($p_B = 0.0036$) in the two CDSs we have written for names A and B. For reasons of simplicity, we assume a fixed loss given default of 60% for both names and a maturity of the basket in $T =$ five years. Figure 5 shows the comparison of a full Monte Carlo simulation of correlated default times with a co-monotonic evaluation of the two assets in the duo basket.

Figure 5 (a) and (b) clearly illustrate the variance reduction effect arising from co-monotonicity. Note that the full simulation approach is based on 200,000 scenarios of three variables (one systematic factor and two idiosyncratic effects) whereas the co-monotonic approach is based on 50,000 scenarios of one uniformly distributed variable.

Figure 5 (c) and (d) illustrate the effect of non-exchangeable exposures. The co-monotonic approach works fine as long as we have the same amount of money at risk for both names. As soon as different obligors cause different losses in case of default, the co-monotonic approach loses the information about ‘who’ is defaulting although the n th-to-default distributions still coincide. In the case of non-exchangeable exposures, it is impossible – based on one single random variable – to correctly address the multinomial character of non-exchangeable exposures (see Bluhm & Overbeck, 2004a) by means of a co-monotonic approach. Therefore, for the co-monotonic approach we have to make an assumption regarding the amount that is lost in the case of the first and second default. Because in the co-monotonic set-up we do not have this information at our disposal we decide for equal ‘exposures at default’ (see figure 5 (d)) for both assets. Alternatively, we could have assumed that the exposure of the asset with higher default probability always constitutes the first loss. No matter what assumptions we make, we have to live with a lack of information in the co-monotonic approach and can only try to overcome this information gap by subjective assumptions leading to an approximation of the truth.

Conclusion

Co-monotonicity is a powerful tool for basket and CDO evaluation because it exactly captures the default timing of a given credit portfolio – even for low-granularity pools – but significantly reduces the simulation effort and simulation noise. However, caution is required if assets have non-exchangeable exposures because in such cases the co-monotonic approach will cause a loss of information regarding the ‘identification by name’ of the first, second and n th loss. Fortunately, the standard situation in basket credit derivatives is the case of equally sized exposures. Our duo basket example (involving different exposure sizes) is somewhat ‘artificial’ and serves for illustration purposes only. Based on the still growing importance of basket credit derivatives and CDOs/CSOs, we believe that the time dependency of defaults and losses needs further investigation in order to develop efficient simulation tools. ■

REFERENCES

- Basel Committee on Banking Supervision, 2004**
International convergence of capital measurement and capital standards
Bank for International Settlements, June
- Bluhm C and L Overbeck, 2004a**
An introduction to CDO modelling and applications
In *Credit Risk: Models and Management*, edited by David Shimko, second edition, Risk Books
- Bluhm C and L Overbeck, 2004b**
Semi-analytic approaches to CDO modeling
Economic Notes 33(2), pages 233–255
- Bluhm C, L Overbeck and C Wagner, 2003**
An introduction to credit risk modeling
Second printing, Chapman & Hall/CRC Financial Mathematics Series
- Finger C, 1999**
Conditional approaches for CreditMetrics portfolio distributions
CreditMetrics Monitor, April
- Finger C, 2000**
A comparison of stochastic default rate models
RiskMetrics Journal, November, pages 49–73
- Gordy M, 2000**
Credit VaR models and risk-bucket capital rules: a reconciliation
Proceedings of the 36th Annual Conference on Bank Structure and Competition, Federal Reserve Bank of Chicago
- Gordy M, 2003**
A risk-factor model foundation for ratings-based bank capital rules
Journal of Financial Intermediation 12(3), July, pages 199–232
- Li D, 1999**
The valuation of basket credit derivatives
CreditMetrics Monitor, April
- Li D, 2000**
On default correlation: a copula function approach
Journal of Fixed Income 6, March, pages 43–54
- Schmidt W and I Ward, 2002**
Pricing default baskets
RISK January, pages 111–114
- Schoenbucher P, 2000**
Factor models for portfolio credit risk
Working paper, December
- Sklar A, 1959**
Fonction de repartition à n dimension et leur marges
Publications de l'Institut Statistique de l'Université de Paris 8, pages 229–231
- Sklar A, 1973**
Random variables, joint distribution functions and copulas
Kybernetika 9, pages 449–460
- Vasicek O, 1991**
Limiting loan loss probability distribution
KMV Corporation, August, see the ‘white paper’ section at www.kmv.com
- Wilde T, 2001**
IRB approach explained
Risk May, pages 87–90

Christian Bluhm is head of credit portfolio management at Credit Suisse in Zurich. Ludger Overbeck is head of portfolio analytics and pricing in the active portfolio management division of corporates and markets at HypoVereinsbank in Munich and professor of mathematics at the University of Giessen. The content of this article reflects the personal view of the authors and does not provide information about the opinion of Credit Suisse or HypoVereinsbank. The authors are grateful for helpful comments and suggestions made by two anonymous referees and Wei Ouyang. Email: christian.bluhm@credit-suisse.com, ludger.overbeck@math.uni-giessen.de