Irreconcilable differences

As Basel has acknowledged, the leading credit portfolio models are equivalent in the case of a single systematic factor. With multiple factors, considerable differences emerge, as Christian Bluhm, Ludger Overbeck and Christoph Wagner demonstrate in this comparison of two well-known models

In the past few years, credit risk modelling and credit portfolio management have attracted increasing interest. Several approaches have been implemented, including CreditMetrics (JP Morgan, 1997), CreditRisk+ (Credit Suisse Financial Products, 1997), PortfolioManager (KMV, 1997) and McKinsey's CreditPortfolioView (Wilson, 1997). Although most of the literature on these was published quite a while ago (see Crouhy & Mark, 1998, for a concise overview), there has still been a need to investigate the differences and agreements between them. Koyluoglu & Hickman (1998), for example, shed some light on the common mathematical roots of some of these models, and Gordy (2000) introduced a "mapping" from CreditMetrics to CreditRisk+ and vice versa.

In this article, we compare the asset-value model in the default-only mode with the actuarial-like model CreditRisk+. The software used for the asset-value model is KMV's PortfolioManager. In the case of CreditRisk+, we used a spreadsheet from Credit Suisse Financial Products (CSFP). Our focus is on risk capital and risk contributions (value-at-risk/CoVAR) and we discuss these issues for uniform and non-uniform portfolios. Due to a completely different modelling of correlations, both frameworks lead to a significantly different estimation and allocation of risk capital.

Uniform portfolios

Both models are treated in the default-only mode, ie, each obligor has only two possible end-of-period states: default and non-default. In the event of default, the lender suffers a loss of fixed size; this is the lender's exposure to the obligor.

□ **KMV's analytical approximation.** Consider a portfolio of m loans with equal exposures. For reasons of simplicity, we neglect recovery rates. Assume that the asset correlation between any pair of different borrowing counterparties equals ρ, and let p be the default probability of any loan in the portfolio. Such a portfolio is called a uniform portfolio. Here, we always assume a time horizon of one year. Default of loan i in the portfolio is indicated by a Bernoulli random variable $L_i \sim B(1, p)$, such that $P[L_i = 1] = p$ and $P[L_i = 0] = 1 - p$. So the variable L_i is the gross loss of the ith loan (recoveries are not taken into account). The portfolio percentage gross loss L^{KMV} is then given by the relative frequency of losses:

$$L^{KMV} = \frac{1}{m}\sum_{i=1}^{m} L_i$$

KMV's analytical approximation of the distribution of L^{KMV} is based on a Merton-type asset-value model where the normally distributed asset returns X_i are correlated according to a one-factor model:

$$X_i = \sqrt{\rho}\,Y + \sqrt{1-\rho}\,Z_i$$

where Y and the Z_is are independent identically distributed standard normal variables. Here, Y represents a common factor, Z_i represents company i's specific effect and ρ is the uniform correlation parameter (systematic risk). It turns out (see Vasicek, 1987, and Finger, 1999) that the limit distribution of L^{KMV} for $m \to \infty$ is given by the so-called normal inverse distribution (not to be confused with the inverse normal distributions that frequently appear in the statistics literature), depending on two parameters, namely the uniform default probability p and the uniform asset correlation ρ, short: $L^{KMV} \sim NID(p, \rho)$. The density of the distribution of L^{KMV} is given by the function:

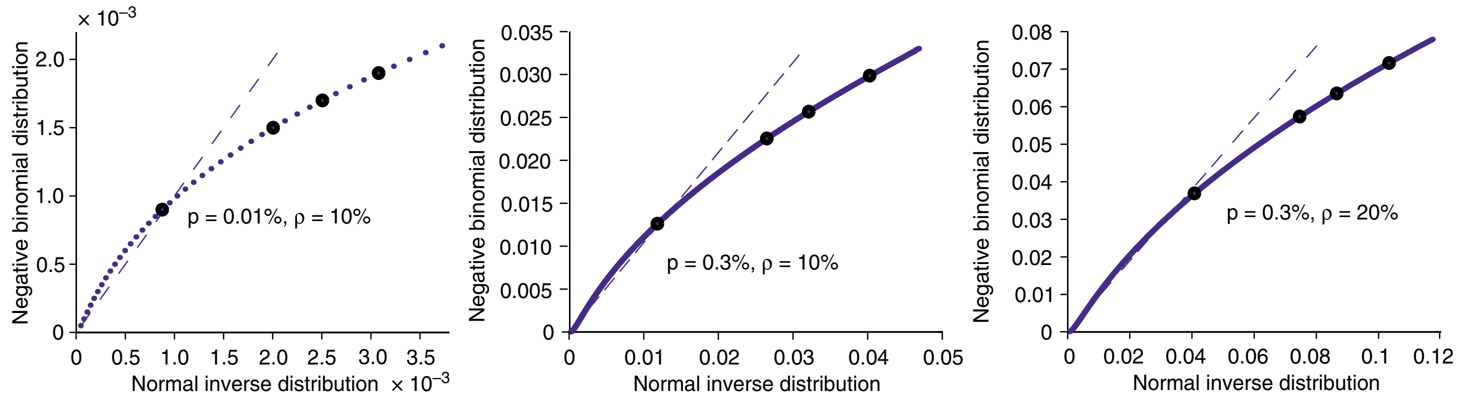
$$\phi_{p,\rho}(x) = \sqrt{\frac{1-\rho}{\rho}}$$

$$\exp\left(-\frac{1}{2\rho}\left((1-2\rho)\left[N^{-1}(x)\right]^2 - 2\sqrt{1-\rho}\,N^{-1}(x)N^{-1}(p) + \left[N^{-1}(p)\right]^2\right)\right)$$

where x is defined as the portfolio loss relative to the total exposure with $0 \le x \le 1$. Here N denotes the standard normal distribution function.

We note that, regarding ρ, there are two extreme cases: $\rho \to 0$ implies $L^{KMV} \sim \delta_p$, where δ_p denotes the Dirac measure concentrated in p, and $\rho \to 1$ leads to $L^{KMV} \sim B(1, p)$. For p, the extreme cases are obvi-

1. Selected Q-Q-plots according to table A



The encircled points show the 99%, 99.9%, 99.95% and 99.98% quantiles respectively; the dashed line represents the diagonal ous: $p \to 0$ yields $L^{KMV} \sim \delta_0$, and for $p \to 1$ one obtains $L^{KMV} \sim \delta_1$.

The first and second moments of the normal inverse distribution are:

$$\mathbf{E}\left[L^{KMV}\right] = p, \qquad \mathbf{V}\left[L^{KMV}\right] = \sigma^2 = N_2\left(N^{-1}(p), N^{-1}(p); \rho\right) - p^2$$

where $N_2(x, y; \rho)$ denotes the bivariate normal distribution function with zero expectation vector and covariance matrix showing units on the diagonal and ρ off the diagonal.

☐ **Loss distribution in CreditRisk+.** CreditRisk+ applies an actuarial framework to derive the loss distribution of a loan portfolio. Contrary to asset-value models, default is not related to the capital structure of the firm. In CreditRisk+, no assumption is made about the causes of default: obligor i is either in default with probability p or is not in default with probability $1 - p$. Let us assume that the single losses $L_i \sim B(1; p)$ are independent. As a consequence, the absolute portfolio gross loss has a binomial distribution, $L^{CR} = \Sigma_{i=1}^{m} L_i \sim B(m, p)$. If m is large and p is small, the portfolio loss L^{CR} approximately has a Poisson distribution with intensity $\lambda = mp$ (CreditRisk+ implicitly assumes infinitely many deals in the portfolio, such that the condition $\lim_{m \to \infty} mp(m) = \lambda$ in the limit yields a Poisson distribution with parameter λ), $L^{CR} \approx P(\lambda)$, $\lambda = mp$. Therefore, the probability of k losses in the portfolio would be:

$$P\left[L^{CR} = k\right] = e^{-\lambda}\frac{\lambda^k}{k!}$$

Now, to replace correlation between asset values, as in the Merton-style model, CreditRisk+ introduces default correlations by assuming default frequencies to be subject to random fluctuations. More precisely, the assumption is that L^{CR} has a Poisson distribution with a gamma-distributed random intensity Λ:

$$\Lambda \sim \Gamma(\alpha, \beta), \qquad \mathbf{E}[\Lambda] = \alpha\beta, \qquad \mathbf{V}[\Lambda] = \alpha\beta^2$$

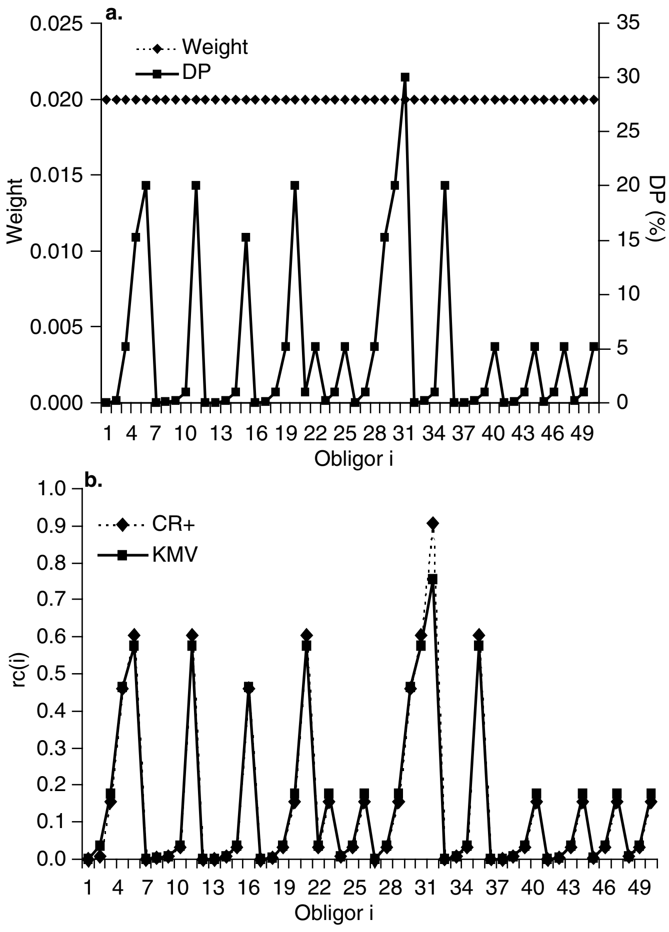
with Γ denoting the gamma function. So, conditional on $\Lambda = \lambda$, L^{CR} admits a Poisson distribution with intensity λ. The unconditional distribution of L^{CR} is then given by the negative binomial distribution (on $\mathbf{N}_0$) with parameters $\alpha\beta$ and α, $L^{CR} \sim NB(\alpha\beta, \alpha)$. This is a standard conclusion from elementary probability (see, eg, Rice, 1995, 8.6.1). Note that based on the properties of the gamma distribution one can interpret α as a shape and β as a scale parameter. Conditional on Λ, defaults of individual obligors are assumed to be independent identically distributed Bernoulli draws, but a high draw of the intensity λ increases the probability of default for each obligor whereas a low draw of λ scales down the default probabilities, thus generating default correlations. This one-factor model easily translates to a multi-factor model. The mean default rate for each obligor is then supposed to be a linear function of gamma-distributed risk factors.

The first and second moments of L^{CR} are $\mathbf{E}[L^{CR}] = \alpha\beta$ and $\mathbf{V}[L^{CR}] = \alpha\beta(1 + \beta)$. Therefore, the distribution of L^{CR} is over-dispersed, ie, the second moment exceeds the first moment.

Note that the CreditRisk+ model does not interpret single losses. Only the total portfolio loss is modelled, more or less driven by the underlying volatility of default rates. This makes a great difference to the KMV model, where every single counterparty is represented by its asset-value distribution.

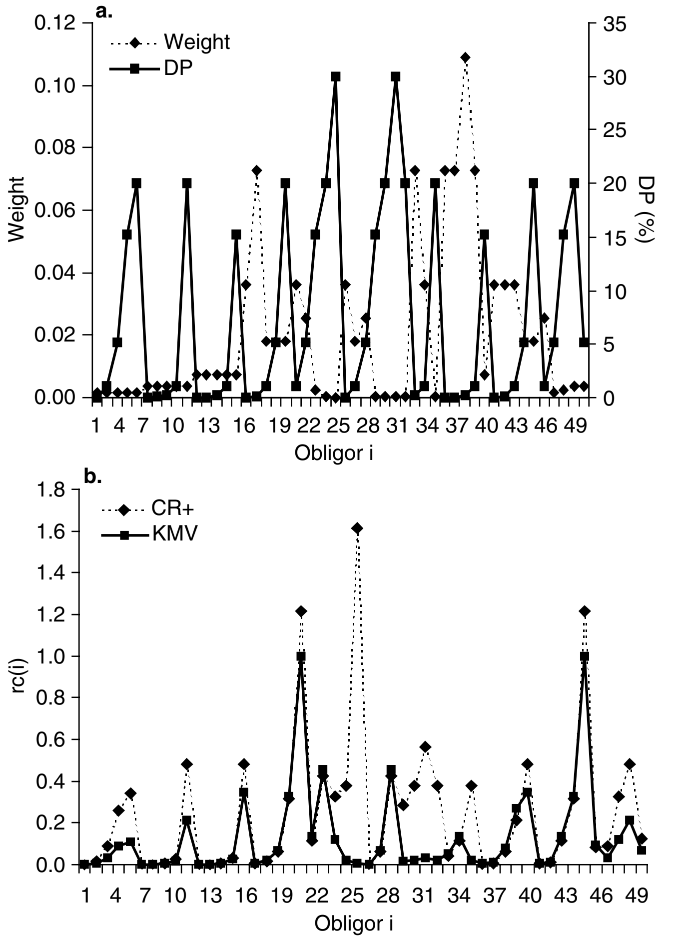
☐ **Framework for comparison.** To compare the two approaches, we

2. Portfolio I



(a) Exposure weights and default probabilities (DP) and (b) risk contributions to the 99.9%-quantile calculated with CreditRisk+ (rescaled) and KMV

3. Portfolio II



(a) Exposure weights and default probabilities (DP) and (b) risk contributions to the 99.9%-quantile calculated with CreditRisk+ (rescaled) and KMV match the first and second moments of the respective loss distributions. More precisely, we fix a (prescribed) default probability p and asset correlation ρ. The portfolio expected loss (EL) and unexpected loss (UL) are given via the respective mean and standard deviation and our constraint now is to choose α, β subject to:

$$\mathbf{E}\left[L^{KMV}\right] = \frac{1}{m}\mathbf{E}\left[L^{CR}\right] = p \quad \text{and} \quad \mathbf{V}\left[L^{KMV}\right] = \frac{1}{m^2}\mathbf{V}\left[L^{CR}\right] = \sigma^2$$

$$\left(m \text{ large, eg, } 20{,}000\right)$$

Setting $m = 20{,}000$ and replacing L^{CR} by L^{CR}/m yields a fairly good approximation of the corresponding percentage loss, because the probabilities $P[L^{CR} = k]$ are negligible for $k \geq 20{,}000$. Therefore, we now have to determine the values of α and β in dependence of p and ρ such that the first and second moments of the two portfolio percentage gross losses agree. According to our constraints, we have two equations to solve for α and β:

$$pm = \alpha\beta \qquad \text{and} \qquad \sigma^2 m^2 = \alpha\beta\left(1+\beta\right)$$

This yields $\alpha = p^2m/(\sigma^2m - p)$ and $\beta = (\sigma^2m - p)/p$, where σ^2 depends on p and ρ. Now it is straightforward to calculate loss distributions for different parameter sets in order to compare the relevant quantiles. This is carried out in the next section.

☐ **Comparison of 99.98%-quantiles and Q-Q-plots.** Table A lists our test cases and the corresponding 99.98%-quantiles. Parameters p, ρ and σ refer to KMV's analytical approximation (A), α and β refer to the negative binomial distribution from the CreditRisk+ model (CR+). In all cases, the 99.98%-quantiles of the CreditRisk+ model are lower than the corresponding ones calculated by analytical approximation, and this effect is most noticeable for small p and ρ. To get the whole picture, we present some corresponding Q-Q-plots in figure 1.

The Q-Q-plots show that in all considered cases the tails of the normal inverse distribution are significantly fatter than the tails of the negative binomial distribution under the constraint of agreement of first and second moments. Following a VAR approach, this would lead to considerable differences in the estimate of risk capital beyond the 99% confidence level.

Non-uniform portfolios

Having treated the uniform case, we are now mainly interested in the behaviour of both models in the opposite case, ie, a rather small number of obligors with variable risk characteristics. For non-uniform portfolios, a purely analytical approach is no longer feasible. We therefore turn now to the portfolio tools of CSFP and KMV and perform some calculations on fictitious sample portfolios with a one-year time horizon and restrict ourselves to four generic test cases. Note that our notation in this section differs slightly from that used above; for consistency with the literature (Credit Suisse Financial Products, 1997), p_i and σ_i now refer to the CR+ model. All our simulations regarding the CR+ framework are based on an Excel spreadsheet provided by CSFP, which unfortunately does not allow VAR calculations for higher quantiles than 99.9%. Within KMV's PortfolioManager, we always use the Monte Carlo simulation to calculate the loss distribution. Drawing 500,000 scenarios guarantees a sufficiently small estimation error up to the 99.98% quantile. Analytical and semi-analytical calculations are also possible but lead to rather poor results for small portfolio sizes such as ours. Furthermore, the parameters of KMV's beta-distributed severity are chosen in such a way that CR+'s loss-given-default $LGD = 1$ is replicated. Since there is no canonical way of translating CR+'s correlation model into the KMV setting and vice versa, we first investigate the extreme cases, ie, no correlations (PF I, PF II), then correlations equal to one (PF III) and then a partly correlated portfolio (PF IV).

☐ **No correlations and equal exposures.** First, we consider a portfolio (PF I) of $m = 50$ obligors with equal exposure weights $w_i = 1/m$ and assign default probabilities p_i (see figure 2a). Following the input parameter setting in the CSFP spreadsheet, CreditRisk+ models the default probability of counterparty i as a random variable with mean p_i and standard deviation σ_i. We choose $\sigma_i = p_i/2$. Next, CreditRisk+ uses a so-called "sector model" with one specific sector and up to seven systematic sectors to incorporate correlations. In our first example, we allow for each obligor i the entire weight on the specific sector, ie, no correlations. On the KMV side, this is imitated by choosing small R-squared, ie, $R_i^2 = 10^{-5}$. Then, the weight of an arbitrarily chosen country and the industry weight of the "non-assigned" category is set to one, the remaining weights are set to zero. Table B shows the EL, the UL (relative to the total exposure) and the 99.9%-quantile of the loss distribution where the length of the confidence in-

A. Parameters and 99.98% quantiles

p	ρ	σ	α	β	$Q_{-[99.98\%]}$ (A)	$Q_{-[99.98\%]}$ (CR)
0.01%	10%	0.02%	0.37	5.38	0.31%	0.19%
0.01%	20%	0.04%	0.08	25.35	0.85%	0.59%
0.01%	30%	0.06%	0.03	78.17	1.67%	1.42%
0.30%	10%	0.35%	0.75	80.25	4.30%	3.14%
0.30%	20%	0.59%	0.26	232.99	9.65%	6.84%
0.30%	30%	0.86%	0.12	496.04	16.69%	12.20%
1.00%	10%	0.96%	1.09	184.32	10.17%	8.11%
1.00%	20%	1.55%	0.42	476.85	20.30%	15.76%
1.00%	30%	2.14%	0.22	911.68	32.17%	25.69%

B. Portfolio I

Portfolio I	EL($1 \cdot 10^{-5}$)	UL($5 \cdot 10^{-4}$)	$Q_{99.9\%}$ ($2 \cdot 10^{-3}$)
CR+	0.0472	0.0308	0.159
KMV	0.0472	0.0281	0.152

C. Credit characteristics of portfolio II

Default prob.	0.01%	0.03%	0.07%	0.20%	1.02%	5.16%	15.25%	20%	30%
Exposure	0.1183	0.1564	0.1127	0.1927	0.2563	0.095	0.0218	0.0469	0.0004
Number	4	5	3	4	10	7	6	9	2
Mean exposure	0.0296	0.0313	0.0376	0.0482	0.0256	0.0135	0.0036	0.0052	0.0002

D. Portfolio II

Portfolio II	EL($1 \cdot 10^{-5}$)	UL($5 \cdot 10^{-4}$)	$Q_{99.0\%}$ ($1 \cdot 10^{-3}$)	$Q_{99.9\%}$ ($2 \quad 10^{-3}$)
CR+, $\sigma = 0.5p$	0.0208	0.0202	0.092	0.136
CR+, $\sigma = p$	0.0208	0.0202	0.092	0.136
CR+, $\sigma = 1.305p$	0.0208	0.0202	0.092	0.136
KMV, $R_i^2 = 10^{-5}$	0.0208	0.0193	0.090	0.134

E. Portfolio III

Portfolio III	EL($1 \cdot 10^{-5}$)	UL($5 \cdot 10^{-4}$)	$Q_{99.0\%}$ ($1 \cdot 10^{-3}$)	$Q_{99.9\%}$ ($2 \cdot 10^{-3}$)
CR+, $\sigma_i = 0.43p_i$	0.0208	0.0224	0.099	0.147
KMV, $R^2 = 0.1$	0.0208	0.0224	0.107	0.164
CR+, $\sigma_i = 1.305p_i$	0.0208	0.0341	0.156	0.255
KMV, $R^2 = 0.4$	0.0208	0.0341	0.171	0.281

tervals for the 90% confidence level are given in brackets (Serfling, 1980).

The small difference in the case of the EL is due to the fact that KMV uses a slightly more sophisticated method (discounting of future cashflows, see KMV, 1997, and Overbeck, 1998). We tried to minimise this effect by choosing the appropriate parameter values. Figure 2b shows the relative risk contributions rc_i of each credit $\Sigma_i\, rc_i w_i / Q_{99.9\%}$, where we have rescaled the CreditRisk+ risk contribution vector by a factor $Q_{99.9\%}(\mathrm{KMV}) = Q_{99.9\%}(\mathrm{CR+})$. The overall agreement between both methods is fairly good.

☐ **No correlations and different exposures.** For our second portfolio (PF II), we allow different exposure weights, assigned to different counterparties according to figure 3a. The rest of the setting remains as for PF I, ie, the default probabilities are kept and we arrive at a portfolio with the following credit characteristics (see table C). Rows two and three of table C show the total exposure and the number of counterparties with a fixed default frequency, and column four gives the exposure per loan. The reason for choosing such an inhomogeneous portfolio is to see how both models perform in this extreme, but uncorrelated, case. The portfolio should therefore also consist of transactions that combine high/low exposure weights and high/low default probabilities. For calculations, we use again only the specific sector (CreditRisk+), corresponding to small R-squared (KMV) and obtain the results shown in table D.

Moments and quantiles show a rather good agreement, but a closer look at the risk contributions (see figure 3b) reveals huge differences for some obligors, especially the ones with small weight and high default probability, eg, obligor $i = 5, 10, 25, 31, 35, 49$ (risk contributions for both frame forks differ by a factor 255 for obligor $i = 25$). It is obvious that, as long as all correlation weight is assigned to the specific sector, CR+'s loss distribution does not depend on σ, since the specific sector is internally modelled as having zero default volatility.

☐ **Maximal correlation within CR+.** For PF III, we use the same exposure weights and default probabilities as for PF II (see figure 3a), but assign within CreditRisk+ all the sector weights to only one systematic sector. This corresponds to maximal (global) correlation. On the KMV side, we try to imitate this by assigning all the country weights to one country and then varying the R^2s. Our constraint here is agreement of first and second moments, and we investigated the two cases $R^2 = 0.1$ and $R^2 = 0.4$.

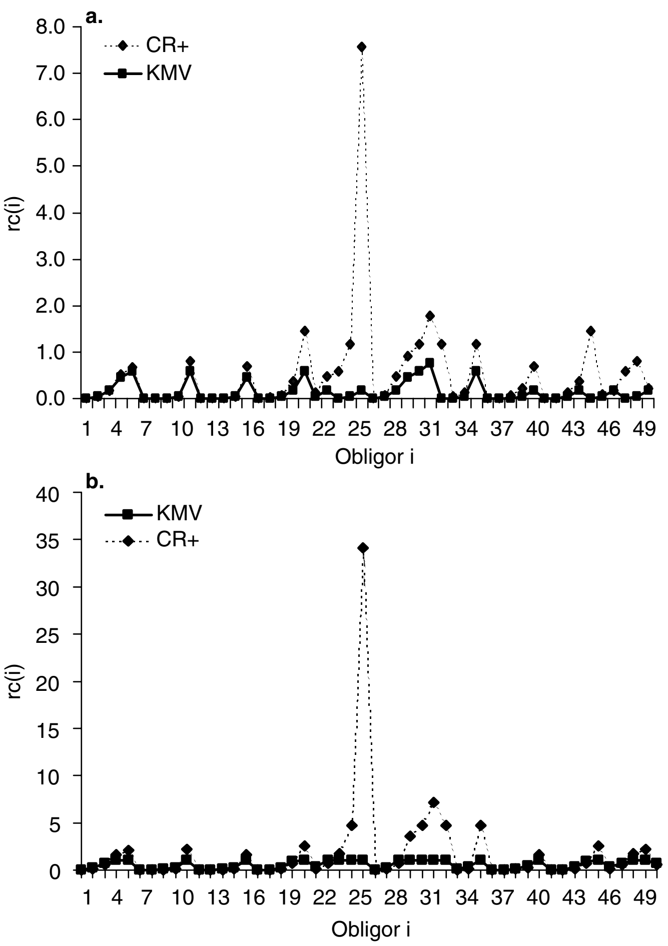
Table E reveals differences in the quantiles well beyond the error bounds. Hence, the tails of KMV's loss distributions are clearly fatter than the ones obtained with CR+ (relative difference up to 10%), implying a difference in the estimation of risk capital of the same order. The risk contributions, plotted in figures 4a and 4b, show great differences (eg, the risk contributions of obligor $i = 25$ differ between the models by a factor of 16 resp. 35 for $R^2 = 0.1$ and $R^2 = 0.4$).

☐ **Partial correlation.** Finally, let us investigate a partly correlated portfolio (PF IV). We start with PF III with $\sigma = 1.305p$, but distribute the CR+ sector weights of obligor i according to figure 5.

First, all weight is allocated to systematic risk factors, ie, no idiosyncratic risk is modelled within the CR+ framework. These correlations are mapped to (i) three different countries (the US, Japan and Germany) and (ii) three different industry factors (agriculture, publishing, semiconductors) within the KMV framework. The results are shown in table F.

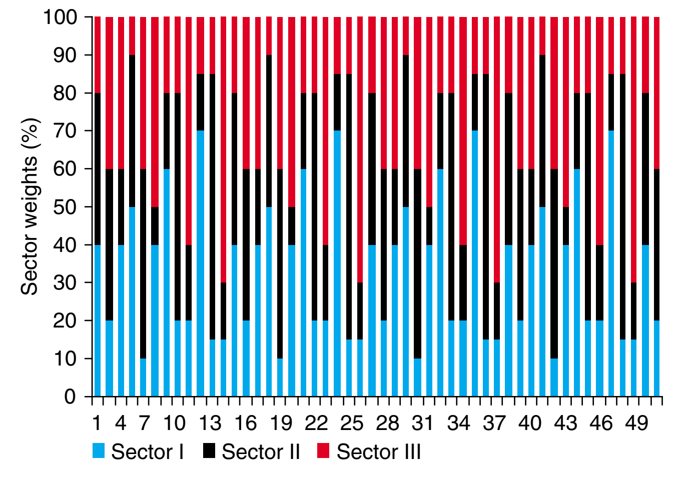
$\sigma_i = 1.305\, p_i$ was chosen in the previous case (PF III, maximal correlation) to match the second moment of the loss distributions. This feature is now completely lost (see first row of table F) as soon as we introduce partial correlation, which also entails huge differences in the quantiles. Even

4. Portfolio III



Risk contributions to the 99.9%-quantile calculated with CreditRisk+ (rescaled) and KMV (a) $R^2 = 0.1$, (b) $R^2 = 0.4$

5. Distribution of sector weights



F. Portfolio IV

Portfolio IV	EL (1 • 10−5)	UL (5 • 10−4)	$Q_{99.0\%}$ (1 • 10−3)	$Q_{99.9\%}$ (2 • 10−3)
CR+, $\sigma_i = 1.305p_i$	0.0208	0.0258	0.115	0.173
CR+, $\sigma_i = 1.305p_i$, idio. risk	0.0208	0.0214	0.096	0.142
CR+, $\sigma_i = 2.15p_i$	0.0208	0.0330	0.151	0.244
KMV (coun.), $R^2 = 0.4$	0.0208	0.0327	0.154	0.278
KMV (ind.), $R^2 = 0.4$	0.0208	0.0333	0.165	0.274

if we match again the unexpected loss by setting $\sigma = 2.15$ (third row) a significant difference in the tail remains. In the previous subsection, we have seen that for full correlation (PF III), a direct identification of the systematic sector with KMV's $R^2 = 1$ would lead to useless results. Nevertheless, and for completeness, we state results (second row of table F) where we set the CR+ weights for the systematic sectors to $w_{i,\,sys} = R_i^2\, w_i$ and $w_{i,\,spec} = 1 - R_i^2$. Here, $w_{i,\,sys}$, $w_{i,\,spec}$ denote systematic and specific sector weights within CR+, and w_i denotes KMV's country/industry weights.[1] Eventually, note that mapping (i) and (ii) yield similar results, which is unsurprising since both categories are treated similarly within KMV's underlying factor model.

Conclusions

In all considered cases, KMV's loss distribution shows fatter tails (and therefore a more conservative estimate of risk capital) than the loss distribution of CreditRisk+. Moreover, it appears to us that the implicit treatment of default correlations in the CreditRisk+ model leads to a weaker representation of correlated tail events than in the asset-value model. Note that Bürgisser *et al* (1999) extended the CreditRisk+ approach by introducing correlation between sectors, but their argument rests on a first and second moment matching, not really changing the weak modelling of tail dependencies within the CreditRisk+ framework. However, an advantage of CreditRisk+ is its simplicity, and the closed form of its loss distribution via probability generating functions is a nice feature for a portfolio model. ■

[1] We also tested this case with Gordy's choice of default volatilities for each rating class (see Gordy), but the results show no substantial difference to the choice $\sigma = 1.305\, p$.

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