Construction of Brownian bridges with self-similar parameter space

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Abstract

The Green's function g on a post-critically finite self-similar set K can be interpreted as covariance kernel of a zero mean Gaussian random field X parametrized by K. In this paper we explicitely construct X by a suitable modification of the well known midpoint displacement algorithm. Our construction includes a random series representation of X, obtained by piecewise harmonic interpolation between Gaussian random variables defined on the vertices of the finite graphs approximating the self-similar set K. For $K = [0, 1]$ our construction yields the Lévy representation of the standard Brownian bridge.

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1 Introduction

In 1987 GOLDSTEIN [8] and KUSUOKA [18] constructed a diffusion process with values in the Sierpinski gasket. Their process is the first example of Brownian motion with values in a self-similar fractal K. For the Sierpinski gasket BARLOW AND PERKINS [3] presented a detailed estimate of the corresponding transition density. Since then many people contributed to the study of diffusions on fractals. For instance, LINDSTRØM [19] studied Brownian motion with values in so-called nested fractals and HAMBLY [9] investigated Brownian motion living in a fractal random environment. All these so called fractional diffusions are strongly symmetric Markov processes, see BARLOW [2] for details. This makes the application of general Markov techniques possible.
The Green's function g associated with Brownian motion in a self-similar set K is a symmetric nonnegative definite kernel. Therefore we can interpret g as covariance kernel of a zero mean Gaussian random field parametrized by the self-similar set K. This viewpoint is due to DYNKIN [4], who established a general theory of Gaussian random fields associated with Markov processes via their potentials, see e.g. [5]-[6]. A decade later, MARCUS AND ROSEN [20]-[21] used DYNKIN's theory to derive continuity and boundedness results for the local times of strongly symmetric Markov processes from the sample path properties of the associated Gaussian fields.
The main novelty presented in this paper is a direct constructive approach to DYNKIN type random fields associated with Brownian motion in suitable self-similar sets K. It will turn out that by construction these random fields are tied down on the boundary of K, which motivates to call them 'bridges'. For the unit interval, our construction yields the well known Lévy representation of the standard Brownian bridge, obtained by the Schauder function construction of Brownian motion. For more general self-similar sets, a basis of piecewise harmonic functions constitutes a substitute for the Schauder functions over the unit interval. As a natural by-product we show that the reproducing kernel Hilbert space of our random fields coincides with the Dirichlet space corresponding to the underlying harmonic structure. For the sake of a better understanding, we explain our ideas in section 2 by means of the unit interval and the Sierpinski gasket before treating the general case in sections 4 and 5. For reasons of convenience for the reader, we recall some basic facts on harmonic calculus and Green's functions in section 3.
For existence of Brownian motion in K we shall make certain assumptions. First, we assume K to be post-critically finite (p.c.f.), essentially meaning that the pieces of the self-similar set K intersect in only finitely many points, see KIGAMI [12] for an exact definition. Second, we assume existence of a harmonic calculus including the notion of a Laplacian and a Dirichlet form on K as introduced by KIGAMI [12]. In general, existence of a harmonic calculus is an open problem, but for many well known fractals such a calculus is well developed and understood, see KIGAMI [11]-[16], MOSCO [22]-[23], STRICHARTZ [25], and STRICHARTZ ET AL. [24], [26].
It is important to point out that KIGAMI's harmonic calculus on p.c.f. self-similar sets yields the same Laplacian and Dirichlet form as derived by general results from the Brownian motion in these sets. So the probabilistic and analytic approaches are just different aspects of the same theory, see BARLOW [2], FUKUSHIMA [7].

2 Brownian bridges over the unit interval and the Sierpinski gasket

In this section we explain our ideas by means of two examples of self-similar sets, namely the unit interval and the Sierpinski gasket. Some background on harmonic calculus and all details and proofs will be given in sections 3, 4, and 5. We first focus on the Brownian bridge $X = \big(X(t)\big)_{t\in[0,1]}$ over the unit interval. Note that $[0,1]$ is the simplest example of a p.c.f. self-similar set. The Lévy representation of X is given by the random series

$$X(t) = \sum_{k=1}^{\infty} \sum_{i\in I(k)} \xi_i^{(k)} S_i^{(k)}(t) \qquad (0 \le t \le 1), \tag{1}$$

where $I(k)$ denotes the set of odd integers between 0 and 2^k, $\{\xi_i^{(k)} : k \in \mathbb{N}, i \in I(k)\}$ is a family of independent standard normal random variables, and the $S_i^{(k)}$'s are taken from the system of Schauder functions obtained by integrating the Haar basis in $L^2([0,1])$. Note that the graphs of the $S_i^{(k)}$'s look like little tents of height $2^{-(k+1)/2}$ centered at $i/2^k$ and nonoverlapping for different values of $i \in I(k)$, see KARATZAS AND SHREVE [10], 2.3.-2.4. The Lévy representation (1) supports convergence of the well known midpoint displacement algorithm for constructing the Brownian bridge on $[0,1]$. The distribution of X is determined by the covariance kernel

$$\Gamma(s,t) = \mathbb{E}\big(X(s)X(t)\big) = \min\{s,t\} - st \qquad \big(s,t \in [0,1]\big).$$

Moreover, Γ is the Green's function of Brownian motion B in $[0,1]$ $\big($killed when it exits $(0,1)\big)$, so that X is the Gaussian field associated with B, see DYNKIN [4].

Now let K be any p.c.f. self-similar set carrying a harmonic structure, see KIGAMI [12]. Note that the standard Laplacian (induced by the underlying harmonic structure) on K corresponds (as infinitesimal generator) to a Brownian motion process[1] B in K. The standard Green's function g of B in K is the covariance kernel of a (DYNKIN type) Gaussian field X parametrized by K. Our main objective is to find a Lévy type construction of X generalizing (1). We now explain how this can be done for the Sierpinski gasket (SG).

Recall that the SG is the unique self-similar set $K \subset \mathbb{R}^2$ satisfying

$$K = F_1(K) \cup F_2(K) \cup F_3(K), \qquad F_i(x) = \frac{1}{2}(x - p_i) + p_i \quad (i = 1,2,3),$$

where $p_1 = (0,0)$, $p_2 = (1,0)$, and $p_3 = \big(1/2, \sqrt{3}/2\big)$ are the vertices of an equilateral triangle of sidelength 1. Set $V_0 = \{p_1, p_2, p_3\}$ and let $\Sigma_m = \{1,2,3\}^m$, $F_\sigma = F_{\sigma_1} \circ \cdots \circ F_{\sigma_m}$ for $\sigma = (\sigma_1, ..., \sigma_m) \in \Sigma_m$, and $\Sigma_* = \bigcup_{m=0}^{\infty} \Sigma_m$. For $\sigma \in \Sigma_m$ we identify the three vertices in $F_\sigma(V_0)$ with the corresponding m-minimal triangle. The set of vertices at stage m is given by $V_m = \bigcup_{\sigma\in\Sigma_m} F_\sigma(V_0)$. Besides we define $V_m^0 = V_m \backslash V_0$. The (finite) set V_0 is understood as the boundary of K. Now let g be the standard Green's function on K, see section 3. $G_1 = g|(V_1^0 \times V_1^0)$ is called the (stage 1) Green's matrix on K, given by

$$G_1 = \frac{1}{10} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}. \tag{2}$$

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space over which all random variables will be defined. Set

$$X(p_i) = 0 \qquad (i = 1,2,3).$$

Choose a Gaussian vector $\boldsymbol{\xi}^{(\emptyset)} = \boldsymbol{\xi} = (\xi_p)_{p\in V_1^0}$ with normal distribution $N(0, G_1)$ (mean 0, covariance G_1). For $\sigma \in \Sigma_m$ we define a Gaussian vector $\boldsymbol{\xi}^{(\sigma)} = (\xi_p^{(\sigma)})_{p\in F_\sigma(V_1^0)}$ with distribution $N\big(0, (3/5)^m G_1\big)$ such that $\big\{\boldsymbol{\xi}^{(\sigma)} : \sigma \in \Sigma_*\big\}$ is a family of independent random vectors. For the meaning of the ratio $3/5$ we refer to (3).

$$X(p) = \xi_p \qquad \text{for } p \in V_1^0$$

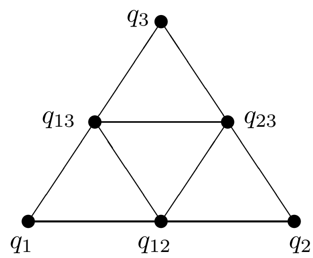
determines X on V_1. We want to extend X by induction to all points $p \in V_* = \bigcup_{m=0}^{\infty} V_m$. Assume that the random variables $X(p)$ are already defined for every $p \in V_{m-1}$. Let $F_\sigma(V_0) = \{q_1, q_2, q_3\}$ be an $(m-1)$-minimal triangle. Then we extend X to $q_{12}, q_{13}, q_{23} \in V_m$ by

[1] Note that one usually constructs a Brownian motion process $\tilde{B}$ in an infinite self-similar extension $\tilde{K}$ of K. Brownian motion in K can then be obtained by killing the process $\tilde{B}$ when it exits K (on the boundary of K).

$$X(q_{12}) = \frac{2}{5}X(q_1) + \frac{2}{5}X(q_2) + \frac{1}{5}X(q_3) + \xi^{(\sigma)}_{q_{12}},$$

$$X(q_{13}) = \frac{2}{5}X(q_1) + \frac{2}{5}X(q_3) + \frac{1}{5}X(q_2) + \xi^{(\sigma)}_{q_{13}},$$

$$X(q_{23}) = \frac{2}{5}X(q_2) + \frac{2}{5}X(q_3) + \frac{1}{5}X(q_1) + \xi^{(\sigma)}_{q_{23}}.$$

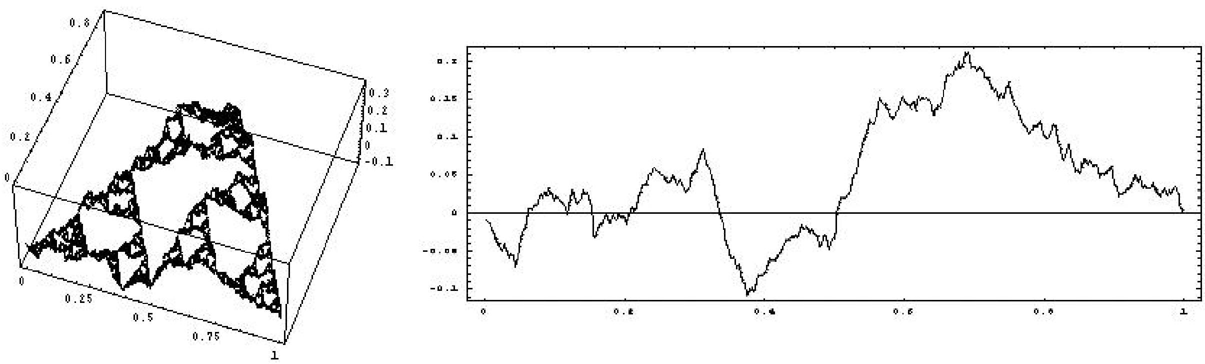


Remark 2.1 The Equations above (without random offsets) are known as the harmonic algorithm, see KIGAMI [11] and STRICHARTZ ET. AL. [24]. This algorithm is used for computing the values of harmonic functions on the SG. More precisely, given any boundary values $f(p_1), f(p_2), f(p_3)$, one can extend the function f to a harmonic function on the SG by successive application of the harmonic algorithm and some limit argument.

If we continue to define random variables by this scheme we obtain a Gaussian system $\{X(p) : p \in V_*\}$ defined on an infinite set of vertices dense in K. It follows from the general results in section 4 and example 3.1 that this system can be extended to the Brownian bridge over K with covariance given by the Green's function g on K, see theorem 4.2.

Moreover, X has a Lévy type representation, see (6). For understanding this, recall that on the unit interval, the Lévy representation (1) is obtained by linear interpolation between random heights arising from midpoint displacement. This is realized by summation of randomly scaled Schauder functions. For the Sierpinski gasket, the Schauder functions are substituted by a basis of piecewise harmonic functions, see example 3.1 for an illustration. By summation of randomly scaled piecewise harmonic function we obtain (6).

The following figure shows a simulation of the Brownian bridge on the SG (left-hand side) and the corresponding front face (right-hand side):



Remark 2.2 A major difference between the bridge over the SG and the bridge over $[0, 1]$ is the following: For the SG we cannot do midpoint displacement with i.i.d. random variables but have to take into account the natural correlations given by the (stage 1) Greens matrix G_1.

3 Green's function on p.c.f. self-similar sets

In this section we briefly recall the construction of Green's function on p.c.f. self-similar sets carrying a harmonic structure. For details and proofs we refer to KIGAMI's papers [12]-[13].

Let $\big(K, \{1, ..., N\}, \{F_i\}_{i=1,...,N}\big)$ be a p.c.f. self-similar structure, essentially meaning that $K = \bigcup_{i=1}^N F_i(K)$ is a compact self-similar set whose pieces $F_i(K)$ intersect in only finitely many points. Let V_0 be the boundary of K, and define $V_m = \bigcup_{\sigma\in\Sigma_m} F_\sigma(V_0)$ for $m \in \mathbb{N}$, where $\Sigma_m = \{1, ..., N\}^m$ and $F_\sigma = F_{\sigma_1} \circ \cdots \circ F_{\sigma_m}$ for $\sigma = (\sigma_1, ..., \sigma_m) \in \Sigma_m$. Note that the p.c.f. property assures that the boundary V_0 is finite. We assume existence of a matrix $D \in \mathbb{R}^{V_0\times V_0}$ and a vector $r = (r_1, ..., r_N)$ with $0 < r_i < 1$, such that (D, r) becomes a (regular) harmonic structure on K, see KIGAMI

[12]-[13] for the definition and properties of harmonic structures. For instance, on the Sierpinski gasket the standard harmonic structure is given by

$$D = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \qquad r = \left(\frac{3}{5}, \frac{3}{5}, \frac{3}{5}\right), \tag{3}$$

see KIGAMI [15] 3.13. Asociated with D is a Dirichlet form $\mathcal{E}_D(u,v) = -u^T Dv$ ($u, v \in \mathbb{R}^{V_0}$). Following KIGAMI's paper [12], any harmonic structure (D, r) induces discrete Laplacians (conductance matrices)

$$H_m = \sum_{\sigma \in \Sigma_m} r_\sigma^{-1} R_\sigma^T D R_\sigma \ \in \mathbb{R}^{V_m \times V_m}$$

and corresponding Dirichlet forms

$$\mathcal{E}_m(u,v) = \sum_{\sigma \in \Sigma_m} r_\sigma^{-1} \mathcal{E}_D(u \circ F_\sigma, v \circ F_\sigma), \qquad \left(u, v \in \mathbb{R}^{V_m}\right)$$

where $r_\sigma = r_{\sigma_1} \cdots r_{\sigma_m}$ and $R_\sigma : \mathbb{R}^{V_m} \to \mathbb{R}^{V_0},\ u \mapsto u \circ F_\sigma$. H_m can be written as

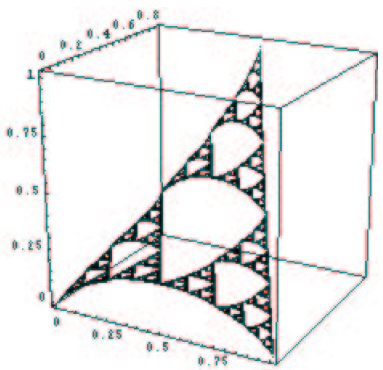
$$H_m = \left(\begin{array}{c|c} T_m & J_m^T \\ \hline J_m & M_m \end{array}\right), \quad T_m \in \mathbb{R}^{V_0 \times V_0}, \quad M_m \in \mathbb{R}^{V_m^0 \times V_m^0}, \tag{4}$$

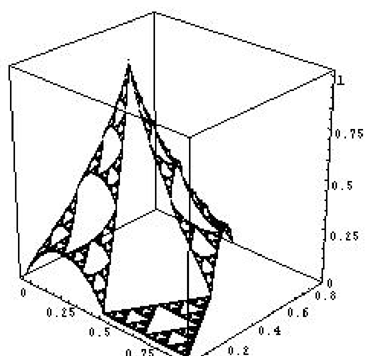
where M_m is invertible and negative definite, so that the so called Green's matrix

$$G_m = -M_m^{-1} \qquad (m \in \mathbb{N})$$

is a symmetric nonnegative definite matrix with nonnnegative entries. Let d denote the Euclidean metric and $C(K) = C(K, d)$ the space of continuous functions (w.r.t. d) on K. A function $f \in C(K)$ is called harmonic on K if $H_m f(p) = 0$ for every $m \in \mathbb{N}$ and all $p \in V_m^0$. A function $f \in C(K)$ is called m-harmonic if $f \circ F_\sigma$ is harmonic for every $\sigma \in \Sigma_m$. For $p \in V_m$ let ψ_p^m be the unique m-harmonic function such that $(\psi_p^m | V_m)(q)$ equals 1 if $q = p$ and 0 otherwise.

Example 3.1 For the Sierpinski gasket K and $p \in V_m^0$ the m-harmonic function ψ_p^m is nonzero on the restriction of K to the two m-minimal triangles connected by p and zero elsewhere. Starting from the characteristic function in p on V_m one can construct ψ_p^m by successive application of the harmonic algorithm (remark 2.1) inside the m-minimal triangles connected by p. The following figure shows ψ_p^0 and ψ_q^1 on the Sierpinski gasket for $p = (1/2, \sqrt{3}/2) \in V_0$ and $q = (1/4, \sqrt{3}/4) \in V_1^0$:





Piecewise harmonic functions like the one on the right-hand side above replace the Schauder functions used in the Lévy representation (1) of the Brownian bridge on the unit interval.

By defining $\psi_p = \psi_p^m$ for $p \in V_m \backslash V_{m-1}$, we obtain a piecewise harmonic basis $(\psi_p)_{p \in V_*}$, where $V_* = \bigcup_{m=0}^{\infty} V_m$. The Green's function at stage 1 is defined by

$$g_1(x,y) = \Psi(x,y) = \sum_{p,q \in V_1^0} (G_1)_{pq} \psi_p(x) \psi_q(y) \quad (x, y \in K).$$

The Green's function at stage m is defined as follows. First set

$$\Psi_\sigma(x,y) = \begin{cases} \Psi\big(F_\sigma^{-1}(x), F_\sigma^{-1}(y)\big) & \text{if } x,y \in F_\sigma(K) \\ 0 & \text{otherwise} \end{cases}$$

for $\sigma \in \Sigma_* = \bigcup_{m=0}^{\infty} \Sigma_m$. Then define

$$g_m(x,y) = \sum_{k=0}^{m-1} \sum_{\sigma \in \Sigma_k} r_\sigma \Psi_\sigma(x,y) = \sum_{p,q \in V_m^0} (G_m)_{pq} \psi_p^m(x) \psi_q^m(y).$$

Because of $\Psi_\sigma(x,y) \geq 0$ (and the regularity of the harmonic structure (D,r)), the limit

$$g(x,y) = \lim_{m\to\infty} g_m(x,y) = \sum_{\sigma \in \Sigma_*} r_\sigma \Psi_\sigma(x,y)$$

is finite, and g is continuous on $K \times K$, see KIGAMI [12] 5.4. The kernel g is called the Green's function associated with the harmonic structure (D,r) on K. It is symmetric and nonnegative definite. More information on Green's functions on fractals and various pictures can be found in a recent paper by KIGAMI, SHELDON, AND STRICHARTZ [17].

4 Construction of Brownian bridges

Let $\big(K, \{1,...,N\}, \{F_i\}_{i=1,...,N}\big)$ be a p.c.f. self-similar structure carrying a regular harmonic structure (D,r) as explained in section 3. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space over which all random variables will be defined. Restriction of the Green's function g to V_1^0 yields the Green's matrix G_1. We mentioned in section 3 that G_1 is a nonnegative definite symmetric matrix with nonnegative entries. Therefore we can interpret G_1 as covariance matrix of a mean zero Gaussian random vector

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{(\emptyset)} = \big(\xi_p^{(\emptyset)}\big)_{p \in V_1^0}.$$

We regard $\xi_p^{(\emptyset)}$ as 'random height' over the vertex $p \in V_1^0$ and define a stochastic process $X^{(0)}$ on K by piecewise harmonic interpolation between the Gaussian variables $\xi_p^{(\emptyset)}$ $(p \in V_1^0)$,

$$X^{(0)}(t) = \sum_{p \in V_1^0} \xi_p^{(\emptyset)} \psi_p(t) \qquad (t \in K),$$

where the functions ψ_p are taken from the piecewise harmonic basis introduced in section 3. Now we select for every address $\sigma \in \Sigma_*$ a Gaussian random vector

$$\boldsymbol{\xi}^{(\sigma)} = \big(\xi_p^{(\sigma)}\big)_{p \in F_\sigma(V_1^0)}$$

with mean zero and covariance matrix $r_\sigma G_1$, such that

$$\Big\{\boldsymbol{\xi}^{(\sigma)} \ : \ \sigma \in \Sigma_*\Big\}$$

is a family of independent random vectors. Recall that for $\sigma \in \Sigma_m$, $r_\sigma = r_{\sigma_1} \cdots r_{\sigma_m}$, where the r_i's are taken from the harmonic structure (D,r). For $m \in \mathbb{N}$ we define processes $X^{(m)}$ by

$$X^{(m)}(t) = \sum_{k=0}^{m} \sum_{\sigma \in \Sigma_k} \sum_{p \in F_\sigma(V_1^0)} \xi_p^{(\sigma)} \psi_p(t) \qquad (t \in K). \tag{5}$$

Proposition 4.1 *The processes $\big(X^{(m)}(t)\big)_{t \in K}$ are zero mean Gaussian random fields with paths in $C(K)$. The covariance kernel Γ_m of $X^{(m)}$ coincides with the Green's function g_{m+1} of stage m+1 associated with the harmonic structure (D,r) on K.*

Proof. First of all it is clear by construction that all random variables $X^{(m)}(t)$ have mean zero. Because all functions ψ_p are continuous, every path of $X^{(m)}$ is just a (finite) linear combination of continuous functions and therefore continuous. For proving that $X^{(m)}$ is Gaussian we first consider only points $p \in V_*$. Every such p is in some V_m. The representation (5) shows that $X^{(m)}(p)$ is a linear combination of variables $\xi_q^{(\sigma)}$. If we group these variables according to their address σ, then the linear combinations inside the groups are Gaussian because the $\boldsymbol{\xi}^{(\sigma)}$'s are Gaussian vectors. The combinations belonging to different addresses are by construction independent. This immediately shows that $\big(X^{(m)}(p)\big)_{p\in V_*}$ is a Gaussian system for every $m \in \mathbb{N}$.
Fix $n \in \mathbb{N}$ and points $t_1, ..., t_n \in K$. We have to verify that $\big(X^{(m)}(t_1), ..., X^{(m)}(t_n)\big)$ is a Gaussian vector. This is the case if and only if $\lambda_1 X^{(m)}(t_1) + \cdots + \lambda_n X^{(m)}(t_n)$ is real Gaussian for every choice of real numbers $\lambda_1, ..., \lambda_n$. But every $t \in K$ is the limit of some sequence $(p_k(t))_{k\geq 1}$ in V_*. We already proved that $\lambda_1 X^{(m)}(p_k(t_1)) + \cdots + \lambda_n X^{(m)}(p_k(t_n))$ must be Gaussian for every $k \in \mathbb{N}$. From this we conclude by an obvious limit argument that $X^{(m)}$ is Gaussian.
The covariance kernel Γ_m of $X^{(m)}$ is given by

$$\Gamma_m(s,t) = \mathbb{E}\, X^{(m)}(s)X^{(m)}(t) = \sum_{k,l=0}^{m} \sum_{\sigma\in\Sigma_k,\,\tau\in\Sigma_l} \sum_{p\in F_\sigma(V_1^0),\,q\in F_\tau(V_1^0)} \Big(\mathbb{E}\,\xi_p^{(\sigma)}\xi_q^{(\tau)}\Big)\,\psi_p(s)\psi_q(t).$$

For $\sigma \neq \tau$ the random vectors $\boldsymbol{\xi}^{(\sigma)}$, $\boldsymbol{\xi}^{(\tau)}$ are independent, which implies $\mathbb{E}\,\xi_p^{(\sigma)}\xi_q^{(\tau)} = 0$. So

$$\Gamma_m(s,t) = \sum_{k=0}^{m} \sum_{\sigma\in\Sigma_k} \sum_{p,q\in F_\sigma(V_1^0)} \Big(\mathbb{E}\,\xi_p^{(\sigma)}\xi_q^{(\sigma)}\Big)\,\psi_p(s)\psi_q(t).$$

For $p, q \in F_\sigma(V_1^0)$ we have by definition of $\boldsymbol{\xi}^{(\sigma)}$

$$\mathbb{E}\,\xi_p^{(\sigma)}\xi_q^{(\sigma)} = (r_\sigma G_1)_{pq}.$$

If we recall the definition of the Green's function via the kernels Ψ_σ, we easily see that

$$\sum_{p,q\in F_\sigma(V_1^0)} (r_\sigma G_1)_{pq}\psi_p(s)\psi_q(t) = r_\sigma \Psi_\sigma(s,t).$$

This implies $\Gamma_m(s,t) = g_{m+1}(s,t)$ on $K \times K$ for every $m \in \mathbb{N}$. ■

The next result is our main theorem.

Theorem 4.2 *As $m \to \infty$, the sequence of functions $t \mapsto X^{(m)}(\omega,t)$ converges uniformly in $t \in K$ to a continuous function $t \mapsto X(\omega,t)$ for a.e. $\omega \in \Omega$. Moreover, the so defined random field X on K is zero mean Gaussian and has a Lévy type representation given by*

$$X(t) = \sum_{\sigma\in\Sigma_*} \sum_{p\in F_\sigma(V_1^0)} \xi_p^{(\sigma)}\psi_p(t) \qquad (t \in K). \tag{6}$$

Its covariance kernel Γ coincides with the Green's function g associated with (D,r) on K. Therefore, if B denotes Brownian motion in K, generated by the Laplacian induced by (D,r), then X is the DYNKIN *type Gaussian field associated with B. According to our terminology, X is the Brownian bridge on K.*

Proof. For $\sigma \in \Sigma_k$ and $p \in F_\sigma(V_1^0)$, the variables $\xi_p^{(\sigma)}$ are normally distributed with mean zero and variance $r_\sigma G_{pp}$. By a standard computation, normality of $\xi_p^{(\sigma)}$ yields

$$\mathbb{P}\left\{|\xi_p^{(\sigma)}| \geq x\sqrt{r_\sigma}\right\} \leq \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{r_\sigma G_{pp}}}{x\sqrt{r_\sigma}}\,\exp\Big(-\frac{r_\sigma x^2}{2\,r_\sigma G_{pp}}\Big) \;\leq\; \frac{\sqrt{c}}{x}\,\exp\Big(-\frac{x^2}{2\,c}\Big),$$

where $c = \max_{p\in V_1^0} G_{pp}$. From this we conclude for every $k \in \mathbb{N}$

$$\mathbb{P}\Big\{\max_{\sigma\in\Sigma_k,p\in F_\sigma(V_1^0)} |\xi_p^{(\sigma)}| \geq k\sqrt{r_\sigma}\Big\} \;=\; \mathbb{P}\Big(\bigcup_{\sigma\in\Sigma_k,p\in F_\sigma(V_1^0)} \Big\{|\xi_p^{(\sigma)}| \geq k\sqrt{r_\sigma}\Big\}\Big) \leq\; N^k(\#V_1^0)\,\frac{\sqrt{c}}{k}\,\exp\Big(-\frac{k^2}{2\,c}\Big).$$

Because of $\sum N^k e^{-k^2/2c} < \infty$, the Borel-Cantelli lemma yields

$$\sum_{k=n(\omega)}^{\infty} \sum_{\sigma\in\Sigma_k} \sum_{p\in F_\sigma(V_1^0)} |\xi_p^{(\sigma)}(\omega)\psi_p(t)| \leq \sum_{k=n(\omega)}^{\infty} \sum_{\sigma\in\Sigma_k} \sum_{p\in F_\sigma(V_1^0)} k\, r^k\, |\psi_p(t)|, \tag{7}$$

where $r = (\max_{i=1,\ldots,N} r_i)^{1/2} < 1$, $n(\omega)$ is some appropriately large index, and $\omega \in \tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$. Now, the p.c.f. property assures that

$$k_0 = \sup_{t\in K} \sup_{k\geq 1} \big(\# \{\sigma \in \Sigma_k \;:\; t \in F_\sigma(K)\}\big) < \infty,$$

see BARLOW [2], 5.21. Based on the structure of the ψ_p's we can estimate (7) from above by

$$\sum_{k=n(\omega)}^{\infty} k_0(\# V_1^0) k\, r^k < \infty.$$

This proves uniform convergence of $X^{(m)}(\omega) \in C(K)$ to $X(\omega) \in C(K)$ as $m \to \infty$, for every $\omega \in \tilde{\Omega}$. A moment's reflection based on the proof of proposition 4.1 shows $\Gamma = g$. ■

5 Dirichlet form and reproducing kernel Hilbert space

In section 3 we defined Dirichlet forms $\mathcal{E}_m$ on V_m, induced by a regular harmonic structure (D, r) on a p.c.f. self-similar set K. As $m \to \infty$, we obtain a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on V_* by

$$\mathcal{F} = \big\{u \in \mathbb{R}^{V_*} : \lim_{m\to\infty} \mathcal{E}_m(u|V_m, u|V_m) < \infty\big\},$$

$$\mathcal{E}(u, v) = \lim_{m\to\infty} \mathcal{E}_m(u|V_m, u|V_m) \quad (u, v \in \mathcal{F}),$$

In this context the natural topology on V_* is given by the effective resistance metric

$$R(p, q) = \min \big\{\mathcal{E}(u, u) \;:\; u \in \mathcal{F},\ u(p) = 1,\ u(q) = 0\big\} \qquad (p, q \in V_*).$$

Because (D, r) was assumed to be regular, the completion of (V_*, R) can be identified with (K, d). More precisely, these metric spaces are homeomorphic and we identify $C(K) = C(K, d)$ and $C\big(\overline{(V_*, R)}\big)$, see KIGAMI [13] 3.1. Additionally we have for $p, q \in V_*$,

$$R(p, q) = \max \big\{|u(p) - u(q)|^2 / \, \mathcal{E}(u, u) \;:\; u \in \mathcal{F},\ u(p) \neq u(q)\big\}. \tag{8}$$

Therefore we can think of $\mathcal{F}$ as a (dense!) subspace of $C(K)$. Now, if μ denotes the uniform self-similar measure on K, then $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F}^0)$ are local regular Dirichlet forms on $L^2(K, \mu)$, where $\mathcal{F}^0 = \big\{f \in \mathcal{F} : f|V_0 = 0\big\}$, see KIGAMI [13] 2.7.

Let X be the Brownian bridge associated with (D, r) on K. Corresponding to the (continuous) covariance kernel $\Gamma : K \times K \to \mathbb{R}$ of X there is a unique Hilbert space $\big(H_\Gamma, \langle\cdot,\cdot\rangle_\Gamma\big)$ in C(K) s.t.

$$\Gamma(p, \cdot) \in H_\Gamma \quad (p \in K), \qquad \big\langle f, \Gamma(p, \cdot)\big\rangle_\Gamma = f(p) \quad (f \in H_\Gamma,\ p \in K),$$

ARONSZAJN [1]. $\big(H_\Gamma, \langle\cdot,\cdot\rangle_\Gamma\big)$ is called the *reproducing kernel Hilbert space* associated with Γ.

Proposition 5.1 *The reproducing kernel Hilbert space H_Γ is given by the Dirichlet space $(\mathcal{E}, \mathcal{F}^0)$.*

Proof. Because the covariance Γ of X is given by the Green's function associated with (D, r), we can apply 7.4.4 in [12] and obtain

$$\mathcal{E}\big(f, \Gamma(x, \cdot)\big) = f(x) \qquad \forall\, f \in \mathcal{F}^0 \quad \forall\, x \in K. \tag{9}$$

Moreover, from [12] 7.4.3 we conclude that

$$\Gamma(x, \cdot) \in \mathcal{F}^0 \qquad \forall\, x \in K. \tag{10}$$

But $(\mathcal{F}^0, \mathcal{E})$ is a Hilbert space ([12] 7.4.5.), so that the assertion follows from the uniqueness of the reproducing kernel Hilbert space associated with Γ. ■

Example 5.2 On $[0,1]$ the standard Laplacian is the second derivative and the standard Dirichlet form is

$$\mathcal{E}(f_1,f_2)=\int_0^1 f_1'(t)f_2'(t)dt \qquad \left(f_1,f_2\in H^1\right),$$

where H^1 is the usual (order 1) Sobolev space in $L^2([0,1])$. The reproducing kernel Hilbert space associated with Γ is given by

$$H_\Gamma=\left\{f\in H^1\ :\ f(0)=f(1)=0\right\}, \quad \left\langle f_1,f_2\right\rangle_\Gamma=\mathcal{E}(f_1,f_2).$$

6 Conclusion

This paper is a step toward stochastic processes with fractal parameter space. It can be seen as an application and concretization of DYNKIN's theory of Gaussian fields associated with Markov processes.

In this paper we were only concerned with the construction and Lévy type representation of Brownian bridges. Many questions remain open. For instance, the picture of the Brownian bridge over the Sierpinski gasket shown in section 2 motivates to study the (irregular) geometry of Brownian bridges over self-similar fractals.

In classical probability theory the most common distribution on the space of continuous functions is the Wiener measure. On (K,D,r) as in our situation, the (Gaussian) distribution P of the Brownian bridge corresponding to (D,r) on K constitutes a substitute for the Wiener measure. This suggests a concept of 'fractal Wiener spaces'. By now we do not know if such a notion produces interesting mathematics, but it is a very natural way to think of fractal analogues of the classical Wiener space.

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