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On a theorem of Kaufman: Cantor-type construction of linear fractal Salem sets

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Abstract. In this paper we present a deterministic Cantor-type construction of linear fractal Salem sets with prescribed dimension. The construction rests on a paper of Kaufman [10] where he investigated the Fourier dimension of the set of α-well approximable numbers in $\mathbf{R}$.

1. Introduction

In metric Diophantine approximation, sets with fractal Hausdorff dimension occur often. As an example we recall the well-known theorem of Jarník [7] and Besicovitch [1], which states that for $\alpha>0$ the set $E(\alpha)$ of α-well approximable numbers in $\mathbf{R}$ has Hausdorff dimension $\dim_H(E(\alpha))=2/(2+\alpha)$. In later years various authors generalized this theorem in many directions (see Dodson [4]). In 1981 Kaufman [10] proved that $E(\alpha)$ carries a probability measure μ_α with compact support whose Fourier transform is of order

$$\hat{\mu}_\alpha(x)=o(\log|x|)|x|^{-1/(2+\alpha)} \quad (|x|\to\infty).$$

By a well-known theorem of Frostman (cf. Mattila [11]) this implies the lower bound in the theorem of Jarník and Besicovitch. Furthermore, it shows that the Hausdorff dimension of the support of μ_α equals its *Fourier dimension*, where the Fourier dimension of a compact set $K\subset\mathbf{R}^d$ is defined by

$$\dim_F(K):=\sup\{\beta\in[0,d]\,|\,\text{there is } \mu\in M_1^+(K) \text{ with } \hat{\mu}(x)=O(|x|^{-\beta/2})\ (|x|\to\infty)\}.$$

Here $M_1^+(K)$ denotes the set of all probability measures with support in K, and $\hat{\mu}$ means the Fourier transform of a measure $\mu\in M_1^+(K)$ defined by

$$\hat{\mu}(x):=\int \exp(-2\pi i\langle x,y\rangle)\,\mu(dy) \quad (x\in\mathbf{R}^d).$$

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A compact set $K \subset \mathbf{R}^d$ is called a *Salem set* if $\dim_F(K) = \dim_H(K)$. The *dimension* of a Salem set means its Fourier resp. Hausdorff dimension. The theorem of Frostman already mentioned implies that the Fourier dimension of compact sets is majorized by their Hausdorff dimension. In certain *random* constructions the occurrence of *fractal* Salem sets seems to be natural (cf. Kahane and Mandelbrot [9], Kahane [8, Chapters 17–18], Salem [12], and Bluhm [3]), but Kaufman's work mentioned above is the only *deterministic* construction of a Salem set of prescribed dimension known to the author. However, his account is by no means easy to follow. In this paper we have tried by modifying his construction and casting it in a more geometric form to produce an easier deterministic construction of linear Salem sets with prescribed dimension. This work is an extended version of Chapter 2 in the author's dissertation [2].

2. Cantor-type constructions

First of all we need some notation. For $x \in \mathbf{R}$

$$\|x\| := \min_{m \in \mathbf{Z}} |x - m|$$

describes the distance from x to the nearest integer. The set of prime numbers will be denoted by $\mathbf{P}$, and we set

$$\mathbf{P}_M := \mathbf{P} \cap [M, 2M]$$

for a positive integer M. Now we explain the Cantor-type construction considered in this paper. Fix $\alpha > 0$ and choose a sequence of positive integers $(M_k)_{k \in \mathbf{N}}$ with

$$M_1 < 2M_1 < M_2 < 2M_2 < M_3 < 2M_3 < \dots .$$

Later we are going to determine recursively a sequence $(M_k)_{k \in \mathbf{N}}$ for which the set

$$S_\alpha := \bigcap_{k=1}^{\infty} \bigcup_{p \in \mathbf{P}_{M_k}} \{x \in [0,1] \mid \|px\| \le p^{-1-\alpha}\}$$

is a Salem set of dimension $2/(2+\alpha)$ (Theorem 3.3). Let us for a moment explain the structure of S_α. For abbreviation we set

$$\overline{E}_q(\alpha) := \{x \in [0,1] \mid \|qx\| \le q^{-1-\alpha}\}$$

for every $q\in\mathbf{N}$. Obviously, $\bar{E}_q(\alpha)$ can be written as a union of closed intervals:

$$\bar{E}_q(\alpha)=[0,q^{-2-\alpha}]\cup\bigcup_{m=1}^{q-1}\left[\frac{m}{q}-q^{-2-\alpha},\frac{m}{q}+q^{-2-\alpha}\right]\cup[1-q^{-2-\alpha},1]. \tag{1}$$

Therefore, the set S_α is compact.

We assume the following Condition 2.1 on $(M_k)_{k\in\mathbf{N}}$ to be fulfilled throughout the paper.

Before formulating it we should recall the prime number theorem in the following form (Hardy and Wright [6, (22.19.3)])

$$\lim_{M\to+\infty}\frac{\#\mathbf{P}_M}{M/\log M}=1, \tag{2}$$

where $\#A$ denotes the number of elements of a finite set A. Therefore, if M_1 is large enough we are able to find a sequence $(M_k)_{k\in\mathbf{N}}$ which fulfills the following condition.

Condition 2.1. Let $M_1\in\mathbf{N}$ be large enough so that

$$\mathbf{P}_{M_k}\neq\emptyset \quad\text{and}\quad \#\mathbf{P}_{M_k}\geq\frac{M_k}{2\log M_k}$$

for every $k\in\mathbf{N}$.

We are now in a position to state the following proposition.

Proposition 2.2. *The set S_α is a nonempty compact set in* $[0,1]$ *and has finite Hausdorff measure for the measure function $h(r)=r^{2/(2+\alpha)}\log(e+r^{-1})$.*

Proof. For proving $S_\alpha\neq\emptyset$ it is sufficient to observe that $0,1\in\bar{E}_p(\alpha)$ for every $p\in\mathbf{P}_{M_k}$ and that $\mathbf{P}_{M_k}\neq\emptyset$ for all $k\in\mathbf{N}$ (Condition 2.1).

For $q\in\mathbf{N}$ the set $\bar{E}_q(\alpha)$ can be covered by $q+1$ intervals of length $2q^{-2-\alpha}$. Once more applying the prime number theorem (2) it is straightforward to show that the set S_α has finite Hausdorff measure for the measure function $h(r)=r^{2/(2+\alpha)}\log(e+r^{-1})$. □

As an immediate consequence we obtain $\dim_F(S_\alpha)\leq\dim_H(S_\alpha)\leq 2/(2+\alpha)$.

Remark 2.3. Closely related to S_α is the set

$$E(\alpha):=\bigcap_{k=1}^{\infty}\bigcup_{q=k}^{\infty}\{x\in[0,1]\mid \|qx\|<q^{-1-\alpha}\}$$

of α-well-approximable numbers. As mentioned in the introduction, the Hausdorff dimension of $E(\alpha)$ is $2/(2+\alpha)$, which was proved by Jarník [7] and Besicovitch [1]. However, the set $E(\alpha)$ is dense in $[0,1]$ and therefore quite different from S_α.

3. Construction of μ_α

In [10] Kaufman constructed a positive measure μ_α with support in $E(\alpha)$ whose Fourier transform is of order

$$\hat{\mu}_\alpha(x)=o(\log|x|)|x|^{-1/(2+\alpha)} \quad (|x|\to\infty).$$

In this section we construct a measure μ_α with a similar decay and support in S_α based on a certain sequence $(M_k)_{k\in\mathbf{N}}$, which will be constructed recursively according to Lemma 3.2 below. The construction rests on a modification of Kaufman's construction. Here $C_c^2(\mathbf{R})$ denotes the space of all twice continuously differentiable functions with compact support.

Before stating Lemma 3.2 we need to introduce some functions. Fix $M\in\mathbf{N}$ with

$$R:=(4M)^{-1-\alpha}<\tfrac{1}{2}, \tag{3}$$

and define a function F_M on $[-\frac{1}{2},\frac{1}{2}]$ by $F_M(x)=\frac{15}{16}R^{-5}(R^2-x^2)^2$ when $|x|\le R$, $F_M(x)=0$ when $R<|x|\le\frac{1}{2}$. In the following we assume F_M to be defined on the whole real line with period 1. Because $F_M\in C^2(\mathbf{R})$ its Fourier series $F_M(x)=\sum_{m\in\mathbf{Z}} a_m^{(M)}e^{2\pi imx}$ converges uniformly to F_M, where the Fourier coefficients $a_m^{(M)}$ are given by

$$a_m^{(M)}=\int_{-1/2}^{1/2}F_M(t)e^{-2\pi imt}\,dt.$$

By (partial) integration we obtain

$$a_0^{(M)}=1,\quad |a_m^{(M)}|\le 1,\quad \text{and}\quad |a_m^{(M)}|\le m^{-2}R^{-2}\quad (m\in\mathbf{N}) \tag{4}$$

for the Fourier coefficients of F_M. Now set

$$q_M(x):=\sum_{p\in\mathbf{P}_M}F_M(px)=\sum_{m\in\mathbf{Z}}\sum_{p\in\mathbf{P}_M}a_m^{(M)}e^{2\pi impx}.$$

Therefore, $q_M\in C^2(\mathbf{R})$ is a 1-periodic function. We intend to normalize q_M by multiplication with a constant c_M^{-1} in order to obtain $c_M^{-1}\hat{q}_M(0)=1$. Because of

$$\hat{q}_M(k)=\sum_{\substack{m\in\mathbf{Z}\\ p\in\mathbf{P}_M\\ k=mp}}a_m^{(M)} \tag{5}$$

it is clear that one has to choose $c_M:=\#\mathbf{P}_M$. For abbreviation we set

$$g_M:=c_M^{-1}q_M.$$

Proposition 3.1. *If $g_M(x)>0$, then there exist $p\in\mathbf{P}_M$ with $\|px\|\le p^{-1-\alpha}$.*

Proof. The function F_M is 1-periodic, which yields to

$$g_M(x)>0 \implies \text{there are } p\in\mathbf{P}_M \text{ and } m\in\mathbf{Z} \text{ with } |px-m|\le R=(4M)^{-1-\alpha}<\tfrac{1}{2}.$$

This implies the assertion of the proposition. □

Roughly spoken, we are going to construct a measure μ_α with support in S_α by repeated multiplication of densities g_{M_k} where $(M_k)_{k\in\mathbf{N}}$ will be chosen recursively according to the following lemma. We introduce the function

$$\theta(x):=(1+|x|)^{-1/(2+\alpha)}\log(e+|x|)\log\log(e+|x|)$$

for the sake of a clearer presentation.

Lemma 3.2. *For every $\psi\in C_c^2(\mathbf{R})$ and $\delta>0$ there exists a positive integer $M_0=M_0(\psi,\delta)$ such that*

$$|[\psi g_M]^\wedge(x)-\hat{\psi}(x)|\le\delta\theta(x)\quad \textit{for } x\in\mathbf{R},$$

for all $M\ge M_0$.

Before proving the lemma we use it for the construction of an appropriate sequence $(M_k)_{k\in\mathbf{N}}$ and a corresponding measure μ_α carried by S_α.

We start with a function $\psi_0\colon\mathbf{R}_+\to\mathbf{R}_+$ with the properties

$$(6)\qquad \psi_0\in C_c^2(\mathbf{R}),\quad \int\psi_0(x)\,dx=1,\quad \psi_0|_{]0,1[}>0,\quad\text{and}\quad \psi_0|_{\mathbf{R}\setminus[0,1]}\equiv 0.$$

Now we choose $0<\tau<\frac{1}{2}$. According to Lemma 3.2 we find

$$\begin{aligned}
&M_1=M_1(\psi_0,\tau 2^{-1}),\\
&M_2=M_2(\psi_0 g_{M_1},\tau 2^{-2}),\\
&\quad\vdots\\
&M_k=M_k(\psi_0 g_{M_1} g_{M_2}\dots g_{M_{k-1}},\tau 2^{-k})\quad (k\in\mathbf{N}).
\end{aligned}$$

We assume S_α to be constructed according to $(M_k)_{k\in\mathbf{N}}$. Now we build products

$$G_0:=1,\quad\text{and}\quad G_k:=\prod_{j=1}^{k} g_{M_j}\quad (k\in\mathbf{N}).$$

Using Lemma 3.2 we obtain for all $k\in\mathbf{N}_0$ and all $x\in\mathbf{R}$

$$|[\psi_0 G_{k+1}]^\wedge(x)-[\psi_0 G_k]^\wedge(x)|\le\tau 2^{-k-1}\theta(x). \tag{7}$$

Denote by λ^1 the Lebesgue measure on the Borel sets in $\mathbf{R}$. Define a sequence of measures by

$$\mu_k:=\psi_0 G_k\lambda^1 \quad (k\in\mathbf{N}_0)$$

with Fourier transforms $\hat{\mu}_k(x)=[\psi_0 G_k]^\wedge(x)$ ($x\in\mathbf{R}$, $k\in\mathbf{N}_0$). Because of (7) the sequence $(\hat{\mu}_k)_{k\in\mathbf{N}_0}$ is a Cauchy sequence with respect to the supremum norm. This implies that

$$\text{there exists } \mu_\alpha\in M_1^+([0,1]) \quad \text{such that} \quad c\mu_k\xrightarrow{w}\mu_\alpha$$

where $\xrightarrow{w}$ denotes weak convergence and $c=c(\tau)$ is a positive constant which normalizes μ_α to mass 1. Now (under assumption of Lemma 3.2) we are able to prove the main theorem of this paper.

Theorem 3.3. *The measure μ_α obeys*

$$\hat{\mu}_\alpha(x)=O(\theta(x)) \quad (|x|\to\infty).$$

Therefore, S_α is a Salem set of dimension $2/(2+\alpha)$.

Proof. The claimed Fourier asymptotic of μ_α follows from (7) and a simple geometric series argument, also taking into account that $\hat{\mu}_p(x)=O(|x|^{-2})$ for fixed p. The second assertion follows from Proposition 2.2, $\dim_F(S_\alpha)\le\dim_H(S_\alpha)$, and Proposition 3.1 (which implies that the support of μ_α is contained in S_α). □

4. Proof of Lemma 3.2

Our task in this section is to prove Lemma 3.2. To begin with, fix $M\in\mathbf{N}$ for a moment. Because of (5) and $|a_m^{(M)}|\le 1$ we have $|\hat{q}_M(k)|\le\#\{(m,p)\in\mathbf{Z}\times\mathbf{P}_M\,|\,k=mp\}$. Because $|k|$ has a unique factorization by prime numbers, we easily obtain

$$|\hat{q}_M(k)|\le\frac{\log|k|}{\log M} \tag{8}$$

for all $k\in\mathbf{Z}\setminus\{0\}$. Additionally, by (4) and (5) we have the implication

$$\begin{aligned} mp=k \quad &\Longrightarrow \quad |m|=\frac{|k|}{p}\ge\frac{|k|}{2M} \quad \Longrightarrow \quad |a_m^{(M)}|\le m^{-2}R^{-2}\le 4k^{-2}M^2R^{-2},\\ &\Longrightarrow \quad |\hat{q}_M(k)|\le\frac{4k^{-2}M^2R^{-2}\log|k|}{\log M} \end{aligned} \tag{9}$$

for all $k\in\mathbf{Z}\setminus\{0\}$. We prove Lemma 3.2 in three steps.

Step 1. *There exists* $M_1>0$ *and* $A=A(\alpha)>0$ *such that for all* $M\geq M_1$,

$$|\hat{g}_M(k)|\leq AM^{-1}\log M \qquad \text{for all } k\in\mathbf{Z}\setminus\{0\},$$
$$|\hat{g}_M(k)|\leq A|k|^{-1/(2+\alpha)}\log|k| \quad \text{for all } k\in\mathbf{Z} \text{ with } |k|>4MR^{-1}.$$

Proof of Step 1. We consider two cases.
Case 1: $1\leq|k|\leq 4MR^{-1}$. Using (8) with Condition 2.1 we have

$$\begin{aligned}|\hat{g}_M(k)|&=c_M^{-1}|\hat{q}_M(k)|\leq\frac{c_M^{-1}\log|k|}{\log M}\leq\frac{2M^{-1}\log M(\log(4M)-\log R)}{\log M}\\&=2M^{-1}(\log 4+\log M+(1+\alpha)(\log 4+\log M))\leq 4(2+\alpha)M^{-1}\log M.\end{aligned}$$

Case 2: $|k|>4MR^{-1}=(4M)^{2+\alpha}$. By (9) we obtain the following estimation,

$$\begin{aligned}|\hat{g}_M(k)|&=c_M^{-1}|\hat{q}_M(k)|\leq\frac{c_M^{-1}4k^{-2}M^2R^{-2}\log|k|}{\log M}\leq\frac{2M^{-1}(\log M)4k^{-2}M^2R^{-2}\log|k|}{\log M}\\&=8k^{-2}MR^{-2}\log|k|=8k^{-2}\tfrac{1}{4}(4M)^{3+2\alpha}\log|k|\leq 2|k|^{-1/(2+\alpha)}\log|k|\end{aligned}$$

It remains to show that $|\hat{g}_M(k)|\leq AM^{-1}\log M$ for $|k|>4MR^{-1}$ for all $M\geq M_1$ with large M_1 and some constant $A=A(\alpha)$. But this is easily verified by combining elementary properties of the logarithm with the estimations in Case 2 (e.g. $A=2(2+\alpha)$). □

From now on let always $M\geq M_1$, and let $\psi\in C_c^2(\mathbf{R})$ be given.

Step 2. *There exists* $B=B(\psi,\alpha)>0$ *such that*

$$|[\psi g_M]^\wedge(x)-\hat{\psi}(x)|\leq BM^{-1}\log M \quad \text{for } x\in\mathbf{R}.$$

Proof of Step 2. Writing g_M as a Fourier series we obtain

$$[\psi g_M]^\wedge(x)=\sum_{k\in\mathbf{Z}}\hat{g}_M(k)\hat{\psi}(x-k).$$

Then, $\psi\in C_c^2(\mathbf{R})$ and $\hat{g}_M(0)=1$ imply

$$\begin{aligned}|[\psi g_M]^\wedge(x)-\hat{\psi}(x)|&\leq\sum_{k\neq 0}|\hat{g}_M(k)||\hat{\psi}(x-k)|\leq B_1\sum_{k\neq 0}|\hat{g}_M(k)|(1+|x-k|)^{-2}\\&\leq B_1\left(\sum_{k\neq 0}(1+|x-k|)^{-2}\right)\sup_{k\neq 0}|\hat{g}_M(k)|\leq BM^{-1}\log M,\end{aligned}$$

with constant $B=B(\psi,\alpha):=2AB_1\sum_{k=1}^{\infty}k^{-2}$, where $A=A(\alpha)$ is the constant from Step 1. Therefore, Step 2 is proven. □

Now let $\delta>0$ be arbitrarily small.

Step 3. *There exists* $M_2>0$ *such that for all* $M\geq M_2$,

$$|[\psi g_M]^\wedge(x)-\hat{\psi}(x)|\leq \delta\theta(x) \quad \textit{for } x\in \mathbf{R}.$$

Proof of Step 3. We consider two cases.

Case 1: $|x|<8MR^{-1}=2(4M)^{2+\alpha}$. In this case the assertion follows for large M from Step 2 and some tedious manipulation.

Case 2: $|x|\geq 8MR^{-1}$. We divide the sum

$$\sum_{k\neq 0} B_1|\hat{g}_M(k)|(1+|x-k|)^{-2}$$

arising in the proof of Step 2 in two parts by summing first over k with $|x-k|\geq\frac{1}{2}|x|$ and second over k with $|x-k|<\frac{1}{2}|x|$. It is easy to see that for large M the first sum is majorized by $C_1|x|^{-1}$ with a constant C_1 independent of M and x. For estimating the second sum we apply Step 1 and use $\frac{1}{2}|x|\geq 4MR^{-1}$ to obtain

$$\begin{aligned}\sum_{|x-k|<|x|/2} B_1|\hat{g}_M(k)|(1+|x-k|)^{-2} &\leq \left(2B_1\sum_{k=1}^{\infty}k^{-2}\right)\sup_{|k|>|x|/2}|\hat{g}_M(k)| \\ &\leq B \sup_{|k|>|x|/2\geq 4MR^{-1}}(|k|^{-1/(2+\alpha)}\log|k|)\leq \delta\theta(x)\end{aligned}$$

for all M which are large enough. □

The assertion of Lemma 3.2 follows by choosing $M_0(\psi,\delta)=M_2$.

5. Conclusions

The proof of Lemma 3.2 shows that the construction of $M_0(\psi,\delta)$ is *explicit*. Therefore, Lemma 3.2 provides a recursive explicit construction of the sequence $(M_k)_{k\in\mathbf{N}}$.

By choosing an appropriate $\alpha>0$ the method of this paper results in an explicit method for constructing linear (fractal) Salem sets with prescribed dimension in $]0,1[$.

It is possible to generalize the results of this paper in two directions.

The first consists in considering a decreasing function $\psi:\mathbf{N}\to\mathbf{R}_+$ instead of the function $q\mapsto q^{-1-\alpha}$. This leads to sets S_ψ (closely related to the set $E(\psi)$ of ψ-well approximable numbers) instead of S_α. Dodson [4] calculated the Hausdorff dimension of $E(\psi)$, and in [2] we proved that S_ψ is a Salem set. The second generalization consists in considering all $x \in \mathbf{R}^d$ with $|x| \in S_\alpha$. Then a paper of Gatesoupe [5] shows that this leads to Salem sets in $\mathbf{R}^d$ (invariant under rotations) with dimensions in $]d-1, d[$ (see [2]).

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